

Definition. A topology on X is a collection $\mathcal{T}$ of $\emptyset$ subsets of X having the properties:

i) $\emptyset, X \in \mathcal{T}$

ii) if $\mathcal{I} \subseteq \mathcal{T}$ then $\bigcup_{T \in \mathcal{I}} T \in \mathcal{T}$ ~~and $\bigcap_{T \in \mathcal{I}} T \in \mathcal{T}$~~

i.e., arbitrary ^{union} intersection of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$

iii) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

~~we say that a subset~~

~~→~~ The pair $(X, \mathcal{T})$ is called a topological space.

→ We say that a subset U of X is open if it belongs to the topology $\mathcal{T}$.

Thus, a topological space is a set X together with a collection $\mathcal{T}$ of its subsets, called open sets, with the properties that both $\emptyset$ & X are in $\mathcal{T}$, arbitrary union & finite intersection of elements of $\mathcal{T}$ is again in $\mathcal{T}$.

→ Let X be a set & $\mathcal{P}(X)$ be its power set (i.e., collection of all the subsets of X). Then $\mathcal{P}(X)$ forms a topology on X and is referred to as discrete topology.

→ $\mathcal{T} = \{X, \emptyset\}$ is also a topology on X & is referred to as indiscrete or trivial topology.

→ Define $\tau_f = \{U \subseteq X : U \text{ s.t. } X-U \text{ is either finite or whole of } X\}$

Then $X-X$ being finite $\Rightarrow X \in \tau_f$

$X-\emptyset = X \Rightarrow \emptyset \in \tau_f$

Now, if $\{U_\alpha\}$ is an indexed family of non-empty elements of τ_f , then

$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ is finite

$\Rightarrow \bigcup U_\alpha \in \tau_f$

Finally, if $U_1, U_2, \dots, U_n$ are non-empty elements of τ_f ,

then $X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$ is finite

$\Rightarrow \bigcap_{i=1}^n U_i \in \tau_f$

Consequently, τ_f defines a topology on X & is referred as finite complement topology.

→ $\tau_c = \{U \subseteq X \mid X-U \text{ is either countable or is all of } X\}$ is a topology on X .

Definition let τ & τ' be two topologies on a given set X . we say τ' is finer (or coarser) than τ if $\tau' \supseteq \tau$. if τ' properly contains τ , we say τ' is strictly finer (or coarser) than τ . If $\tau' \not\supseteq \tau$ & $\tau \not\supseteq \tau'$,

→ if either $\tau' \supseteq \tau$ or $\tau \supseteq \tau'$, we say τ is comparable with τ' .

→ other words such as larger & smaller are also used or stronger or weaker

1. Basis for a topology is a collection $\mathcal{B}$ of subsets of X (called basis elements) s.t.
- for each $x \in X$, $\exists$ atleast one $B \in \mathcal{B}$ s.t. $x \in B$
 - if $x \in B_1 \cap B_2$; $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

→ Topology τ generated by a basis $\mathcal{B}$ is defined as;
 A subset U of X is said to be open in X (i.e., to be an element of τ), if for each $x \in U$, $\exists$ a $B \in \mathcal{B}$ s.t. $x \in B$ & $B \subset U$.

Note - Each basis element is itself an element of τ .

Examples (of Basis):-

- 1) The ~~set~~ ^{collection} of all circular regions in a plane.
- 2) The ~~set~~ ^{collection} of all rectangular regions in a plane.
- 3) If X is any set then ~~set~~ ^{collection} of all one-point subsets of X (i.e., singleton subsets of X) forms a basis for the discrete topology on X .

Result:- The collection of all possible unions of elements of a basis $\mathcal{B}$ is equal to the whole topology τ .

→ Every element of τ can be expressed as union of basis elements (not necessarily unique).

Note

Let X be an ordered set in ordered topology & $Y \subseteq X$ is convex in X . Then the order topology on Y is the same as the topology Y inherits as a subspace of X .

→ In general, it is not true.

e.g., $Y = \{1\} \cup \{2\} \subseteq \mathbb{R}$

Then $\{2\}$ is open in Y in the subspace topology on Y inherited from $\mathbb{R}$ but it is not open in the ordered topology on Y as any ^{basic} element of the order topology on Y containing 2 is of the form

$$\{x \mid x \in Y \text{ \& \> } a \text{ \& \< } 2\}$$

from subspace topology $\{2\} = Y \cap \left(\frac{3}{2}, \frac{5}{2}\right)$ ↳ such a set always contains points less than 2 of Y .

→ Let Y be a subspace of X & $A \subseteq Y$ then the topology A inherits as subspace of Y is same as the topology it inherits as subspace of X .

~~set~~

Closed sets A subset A of X is said to be closed if the set $X - A$ is open.

→ $\emptyset, X$ are closed

→ Arbitrary ^{Intersection} of closed sets is closed.

→ Finite union of closed sets is closed.

→ Let Y be a subspace of X , then a set A is closed in Y iff it equals the intersection of a closed set in X with Y . i.e., $A = C \cap Y$, C is closed in X .

- Let $\pi_1 : X \times Y \rightarrow X$ as $\pi_1(x, y) = x$;
- $\pi_2 : X \times Y \rightarrow Y$ as $\pi_2(x, y) = y$.
- Then, π_1 & π_2 are called projections of $X \times Y$ onto its first & second factors, resp.
- if $U \subseteq X$ is open then $\pi_1^{-1}(U) = U \times Y$
- & if $V \subseteq Y$ is open then $\pi_2^{-1}(V) = X \times V$
- Then, $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V$
- $\mathcal{S} := \{ \pi_1^{-1}(U) \mid U \subseteq X \text{ is open} \} \cup \{ \pi_2^{-1}(V) \mid V \subseteq Y \text{ is open} \}$ is a subbasis for the product topology on $X \times Y$.
- Let (X, τ) be a topological space & $Y \subseteq X$.
- Define $\tau_Y := \{ Y \cap U \mid U \subseteq X \text{ is open (i.e., } U \in \tau) \}$.
- Then (Y, τ_Y) is also a topological space.
- τ_Y is called subspace topology & (Y, τ_Y) as subspace of (X, τ) .
- if $\mathcal{B}$ is a basis for the topology on X , then the collection $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for subspace topology.
- if A & B are resp the subspaces of X & Y then the product topology on $A \times B$ is same as the topology $A \times B$ inherits as a subspace of $X \times Y$.
- $Y \subseteq X$ is said to be connected if for $a, b \in Y$ the interval of points (a, b) of X lies in Y .
- Note: Interval & arcs in X are connected.

$\hookrightarrow B = \{ (a \times b, a \times d) ; a \times b, a \times d \in \mathbb{R} \times \mathbb{R} \}$ is also a basis for the order topology on $\mathbb{R} \times \mathbb{R}$.

$\hookrightarrow$ The order topology on $\mathbb{Z}_+$ is the discrete topology, for every ~~point~~ one-point set is open: if $n > 1$, then $\{n\} = (n-1, n+1)$ is a basis element; & if $n=1$, $\{1\} = [1, 2)$ is a basis element.

$\hookrightarrow X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order is ~~discrete~~ an ordered set with smallest element. The order topology on X is not discrete.

$\hookrightarrow$ Rays determined by a

$\left\{ \begin{array}{l} \text{Rays determined by } a \end{array} \right.$	$\left\{ \begin{array}{l} (-\infty, a) = \{x \mid x < a\} \\ [a, +\infty) = \{x \mid x \geq a\} \end{array} \right.$	$\left. \begin{array}{l} (-\infty, a) = \{x \mid x < a\} \\ [a, +\infty) = \{x \mid x \geq a\} \end{array} \right\} \begin{array}{l} \text{open rays} \\ \text{closed rays} \end{array}$

Definition Let X & Y be topological spaces. Then the product topology on $X \times Y$ is the topology having basis as the collection B of all ~~sub~~ sets of the form $U \times V$, where U is open subset of X & V is an open subset of Y .

$\hookrightarrow$ if B is a basis for X & C is a basis for Y then $D := \{ A \times B \mid A \in B \text{ & } B \in C \}$ is a basis for (product) topology on $X \times Y$.

Definition 1 A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X . The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection τ of all the unions of finite intersections of elements of $\mathcal{S}$.

→ That is, $\mathcal{B} = \{ B \subseteq X \mid B = \bigcap_{\text{finite}} S_i ; S_i \in \mathcal{S} \}$ forms a basis for τ .

→ $(X, <)$ be a simply ordered set then

~~$\mathcal{B} = \{ (a, b) : a, b \in X \}$~~

Let $\mathcal{B}$ be a basis for a topology on X defined by the following intervals:

- $(a, b) = \{ x \mid a < x < b \}$ → (a, b) open interval
- $[a, b) = \{ x \mid a \leq x < b \}$ → (a, b) half-open intervals
- $(a, b] = \{ x \mid a < x \leq b \}$
- $[a, b] = \{ x \mid a \leq x \leq b \}$ → $[a, b]$ closed interval in X .

Let $\mathcal{B} = \{ (a, b) \mid a, b \in X \} \cup \{ [a_0, b) : a_0 \text{ is smallest elt of } X \text{ (if it exists)} \}$

$\{ (a, b_0] \mid a, b_0 \in X \text{ where } b_0 \text{ is the largest element of } X \text{ (if it exists)} \}$.

Then $\mathcal{B}$ forms a basis for a topology on X , which is called order topology.

Example 1 Standard topology on $\mathbb{R}$ with usual order on $\mathbb{R}$.

→ $\mathbb{R} \times \mathbb{R}$ with dictionary order i.e., if we denote an elt. of $\mathbb{R} \times \mathbb{R}$ as $a \times b$; $a, b \in \mathbb{R}$ then, for $a \times b, c \times d \in \mathbb{R} \times \mathbb{R}$, $a \times b < c \times d$ iff either $a < c$ or $a = c$ & $b < d$.

$\rightarrow \tau'$ is finer than τ (ie, $\tau' \supset \tau$) iff, for each $x \in X$ & each $B \in \mathcal{B}$ containing x , there is a $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$ (where $B \xrightarrow{\text{basis}} \tau$, $B' \xrightarrow{\text{basis}} \tau'$).

$\rightarrow$ The collection of all circular region & rectangular regions in the plane generate the same topology.

$\rightarrow$ Collection of all the open intervals in the real line forms a topology basis for a topology on $\mathbb{R}$ known as standard topology on real line.

$\rightarrow$ Consider the collection $\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}\}$ where $[a, b) = \{x \mid a \leq x < b\}$. Then the topology generated by $\mathcal{B}'$ is called lower limit topology on $\mathbb{R}$ & then we denote $\mathbb{R}$ with this topology as $\mathbb{R}_l$.

$\rightarrow$ Define $K := \{\frac{1}{n} \mid n \in \mathbb{N}\}$ & let $\mathcal{B}'' = \{(a, b) \mid a, b \in \mathbb{R}\} \cup \{(a, b) - K \mid a, b \in \mathbb{R}\}$.

Then, the topology generated by $\mathcal{B}''$ will be called K -topology on $\mathbb{R}$ & then we denote $\mathbb{R}$ with this topology as $\mathbb{R}_K$.

$\rightarrow \mathbb{R}_l$ & $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ & are not comparable with each other.

- Let Y be a subspace of X . If A is closed in Y & Y is closed in X then A is closed in X .
- (In general, it is not true)

Interior of A : Union of all the open sets contained in A → denoted by $\text{Int } A$, then

Closure of A : Intersection of all the closed sets containing A .
denoted by $\text{cl } A$ or $\bar{A}$, closed

If A is open, then $\text{Int } A = A$

If A is closed, then $\bar{A} = A$.

Let Y be a subspace of X & $\bar{A}$ denoted the closure of $A \subseteq Y$ in X , then closure of A in Y is $\bar{A} \cap Y$.

$x \in \bar{A} \Leftrightarrow$ every nbd containing x intersects A .
 $x \in \bar{A} \Leftrightarrow$ every open set containing x intersects A .
 $x \in \bar{A} \Leftrightarrow$ every basis elt. containing x intersects A .

$x \in X$ is said to be a limit point (or cluster point or point of accumulation) of $A \subseteq X$ if every nbd containing x intersects A in some pt other than x itself.

OR x is a pt of A if it belongs to $A - \{x\}$.

The closure of A is $\bar{A} = A \cup A'$ where A' is the set of all limit points of A .

$A \subseteq X$ is closed iff it contains all its limit pts.

→ The box topology on $\prod X_\alpha$ has as basis all the sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α .

→ The product topology on $\prod X_\alpha$ has as basis all the sets of the form $\prod U_\alpha$, where U_α is open in X_α and U_α equals X_α except for finitely many values of α .

§ Box topology is generally finer than product topology.

→ If $\mathcal{B}_\alpha$ is given as basis for X_α then the collection of set of the form $\prod B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ will serve as ~~box topology~~ basis for box topology.

§ the collection of all sets of the form $B_\alpha \times B_\alpha$ for finitely many indices α & $B_\alpha = X_\alpha$ for all the remaining indices will serve as basis for product topology.

→ If A_α be subspace of X_α for each α . Then, $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ provided both products are given to be either box topology or product topology.

→ If each X_α is Hausdorff then $\prod X_\alpha$ is also Hausdorff (in both box & product topology)

→ Let $A_\alpha \subset X_\alpha$ for each α . Then if $\prod X_\alpha$ are given to be either box topology or product topology then $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

$\hookrightarrow$ Consider the Cartesian product $X_1 \times X_2 \times \dots \times X_n$ and $X_1 \times X_2 \times \dots$ of topological spaces X_i 's.
 Let U_i be an open set of X_i for each i .
 Then defining all the sets of the form $U_1 \times U_2 \times \dots \times U_n$ and $U_1 \times U_2 \times \dots$ resp. as basis forms defines a topology on the Cartesian product called box topology.

$\hookrightarrow$ Let $\pi_i: X_1 \times X_2 \times \dots \times X_n / X_1 \times X_2 \times \dots \rightarrow X_i$ be the projection. We take $\pi_i^{-1}(U_i)$; i is any index & U_i is open set of X_i as subspace. Then the topology defined in this way is called product topology.

$\hookrightarrow$ If $\{X_\alpha\}_{\alpha \in I}$ is an indexed family of topological spaces & U_α is open in X_α then taking πU_α as basis elements forms the box topology on the product space $\prod_{\alpha \in I} X_\alpha$.

$\hookrightarrow$ Let $S_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$ & let $S = \bigcup_{\beta \in I} S_\beta$.
 The topology generated by the subspace S is called the product topology. In this case, $\prod_{\alpha \in I} X_\alpha$ is referred as product space.

→ If condition (iv) holds, we say f is cont. at $x \in X$.

Homeomorphism: if both f & f^{-1} are continuous where $f: X \rightarrow Y$ is a bijection.

f preserves the topological structure involved.

→ Suppose $f: X \rightarrow Y$ is an injective cont map.

Let $Z = f(X)$ considered as subspace of Y .

then the function $f': X \rightarrow Z$ is bijective.

If f' is homeomorphism then $f: X \rightarrow Y$

is a (topological) Imbedding of X in Y .

→ If $X = A \cup B$ where A & B are closed in X .

Let $f: A \rightarrow Y$ & $g: B \rightarrow Y$ be cont. if $h: X \rightarrow Y$ defined

$$h(x) = g(x) \quad \forall \quad x \in A \cap B$$

$$\text{as } h(x) = f(x) \quad \text{if } x \in A$$

$$= g(x) \quad \text{if } x \in B$$

is cont.

→ Pasting Lemma

→ $f: X \times A \rightarrow X \times Y$ be s.t. $f(a) = (f_1(a), f_2(a))$

Then f is cont. iff $f_1: A \rightarrow X$ & $f_2: A \rightarrow Y$ are cont.

Subspace of a Hausdorff space is Hausdorff. 6

Continuous function

Let X & Y be topological spaces. $f: X \rightarrow Y$ is said to be continuous if for each open subset V of Y , $f^{-1}(V)$ is an open subset of X .

f is continuous if inverse image of every basis element of Y is open.

(as if V is an arbitrary open set of Y then $V = \bigcup_{\alpha \in J} B_{\alpha}$; B_{α} are basis elements)

$$\therefore f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_{\alpha})$$

Let S be a subspace for Y , then $f: X \rightarrow Y$ is continuous if $f^{-1}(B)$; $B \in S$, is open in X .

since, if B is an arbitrary basis element, then $B = S_1 \cap \dots \cap S_n$; $S_i; 1 \leq i \leq n \in S$.

Result Following are equivalent:

i) f is continuous

ii) $f(\bar{A}) \subset \overline{f(A)}$

iii) For every closed set B of Y , $f^{-1}(B)$ is closed

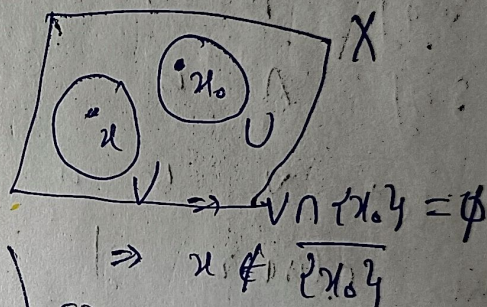
iv) For each $x \in X$, and nbd V of $f(x)$, $\exists$ a nbd U of x s.t. $f(U) \subset V$.

Definition A topological space X is said to be Hausdorff space τ_f for each pair $x_1, x_2 \in X$ where $x_1 \neq x_2$, $\exists$ neighbourhoods U_1 & U_2 of x_1 & x_2 , resp, s.t. $U_1 \cap U_2 = \emptyset$.

i.e., for every distinct pair of points there exists disjoint neighbourhoods containing the pair resp.

$\hookrightarrow$ Every one-point set of Hausdorff space X is closed.

consequently every finite point set of X is closed



$\hookrightarrow (\mathbb{R}, \tau_f)$ is not Hausdorff. $\left\{ \begin{array}{l} \text{Since } x \in X \text{ is arbitrary} \\ \Rightarrow \overline{\{x_0\}} = \{x_0\}. \end{array} \right.$

T_1 -Axiom of ~~in~~ a topological space the condition that finite point sets are closed.

$\hookrightarrow$ Let X be satisfying T_1 -axiom & $A \subseteq X$, then x is a lt pt of A iff every nbd of x contains infinitely many points of A .

$\hookrightarrow$ Every simply ordered spaces in product topology is Hausdorff. ~~on product~~

$\hookrightarrow$ Product of two Hausdorff spaces is Hausdorff.

Metric Topology

If d is a metric on X , then the collection of all ϵ -balls $B_d(x, \epsilon)$, for $x \in X$ & $\epsilon > 0$, is a basis for a topology on X , called the metric topology induced by d .

$\hookrightarrow$ A set U is open in the metric topology induced by d iff for each $y \in U$, there is a $\delta > 0$ s.t. $B_d(y, \delta) \subset U$.

Definition A topological space X is said to be metrizable if $\exists$ a metric d on the set X that induces the topology of X .

$\hookrightarrow$ Let (X, d) be a metric space. Define $\bar{d}: X \times X \rightarrow \mathbb{R}$ by $\bar{d}(x, y) = \min\{d(x, y), 1\}$.

Then $\bar{d}$ is a metric that induces the same topology as d .

$\S$ $\bar{d} \rightarrow$ standard bounded metric corr. to d .

$\hookrightarrow$ Euclidean metric $\rightarrow d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2]^{1/2}$
where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
 $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

$\S$ Square metric $\rightarrow \rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$

$\S$ The topologies induced on $\mathbb{R}^n$ by the Euclidean metric d & square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Definition Given an index set J & point $x = (x_\alpha)_{\alpha \in J}$ & $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{p}$ on $\mathbb{R}^J$ as $\bar{p}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$; $\bar{d}$ is standard bounded metric on $\mathbb{R}$.

Then, $\bar{p}(x, y)$ is called uniform metric & the topology induced by it is called uniform topology.

Definition: Let $f_n: X \rightarrow Y$ be a seq. of functions from the set X to the metric space Y . Let d be the metric for Y . We say (f_n) converges uniformly to the function $f: X \rightarrow Y$ if given $\epsilon > 0$, $\exists$ integer N s.t. $d(f_n(x), f(x)) < \epsilon \quad \forall n > N \quad \& \quad \forall x \in X$.

Uniform Limit Theorem: Let $f_n: X \rightarrow Y$ be a seq. of cont. functions from the topological space X to the metric space Y . If (f_n) converges uniformly to f , then f is continuous.

$\hookrightarrow$ Consider $\mathbb{R}^X = \{f \mid f: X \rightarrow \mathbb{R}\}$ with uniform metric $\bar{P}$. Then, the seq. of functions $f_n: X \rightarrow \mathbb{R}$ uniformly converge to $f: X \rightarrow \mathbb{R}$ iff in the seq. (f_n) converge to f when they are considered as elements of the metric space $(\mathbb{R}^X, \bar{P})$. ~~It is so~~ ~~prove~~ ~~prove~~

$\hookrightarrow$ An uncountable product of $\mathbb{R}$ with itself is not metrizable.

$\hookrightarrow \mathbb{R}^{\omega}$ in box topology is not metrizable.

$\hookrightarrow p: X \rightarrow Y: X, Y \rightarrow$ topological spaces, is said to be quotient map provided ~~a subset~~ $U \subset Y$ is open iff $p^{-1}(U)$ is open in X . $\rightarrow$ aka strong continuity.

$\hookrightarrow$ A subset $C \subset X$ is saturated (w.r.t onto map $p: X \rightarrow Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. i.e., either $p^{-1}(\{y\}) \subset C$ or $p^{-1}(\{y\}) \cap C = \emptyset$.

$\Rightarrow C = \bigcup_{y \in Y} p^{-1}(\{y\}) = p^{-1}(S \subseteq Y)$ i.e., C is saturated if it equals complete inverse image of a subset of Y .

→ First Countable Product of metrizable topologies is metrizable.

→ Continuity of f is equivalent to ϵ - δ function (asymmetrically)

→ The Sequence Lemma: Let X be a topological space let $A \subset X$. If there is a seq of points of A converging to x , then $x \in \bar{A}$; the converse holds if X is metrizable.

→ Let $f: X \rightarrow Y$. If f is cont, then for each seq $x_n \rightarrow x$ in X , $f(x_n) \rightarrow f(x)$. The converse is true if X is metrizable.

→ A space X is said to have a countable basis at the point x if there is a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of nbds of x s.t. any nbd V of x contains at least one of the sets U_n .
A space X that has a countable basis at each of its points is said to satisfy the first countability axiom.

→ A metrizable space is always satisfies first countability axiom.

→ $+$, $-$, $\cdot$ are cont fun $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$
 $\div$ is cont fun $\mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$

→ $f, g: X \rightarrow \mathbb{R}$ are cont, then $f+g, f-g, f \cdot g, f/g$ (if $g(x) \neq 0 \forall x$) are also cont.

Choose an ϵ -ball $B_d(p, \epsilon)$ lying in U .
 Again, let V be an open set in the basis element of dictionary order topology

$$V = (x, y + \epsilon)$$

$$V = ((x, y - \epsilon), (x, y + \epsilon)) = (x \times \overline{y - \epsilon}, x \times \overline{y + \epsilon})$$

if $q = (x_1, y_1) \in \mathbb{R} \times \mathbb{R}$ then $(x, y) = (x_1, y_1)$

$$d((x, y), (x_1, y_1)) = \begin{cases} 0 & \text{if } x \neq x_1 \\ 1 & \text{if } x = x_1 \text{ \& } y \neq y_1 \\ |y - y_1| & \text{if } x = x_1 \text{ \& } y = y_1 \end{cases}$$

if $q \in V$, then

$$d(p, q) < \epsilon \Rightarrow q \in B_d(p, \epsilon) \Rightarrow V \subset B_d(p, \epsilon)$$

Conversely, let $U = (a, b) \times (c, d)$ be an open set in the dictionary order topology s.t. $p = (x, y) \in U$.

Claim: $p \in B_d(p, \epsilon) \subset U$.

Let $q \in B_d(p, \epsilon)$

Consider the open intervals $(x - \epsilon_1, x + \epsilon_1)$ centered at x and $(y - \epsilon_2, y + \epsilon_2)$ centered at y lying in U respectively.

III) Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$, $R = (x_3, y_3) \in \mathbb{R} \times \mathbb{R}$, we need to show that

$$d(P, R) \leq d(P, Q) + d(Q, R)$$

Case I $x_1 = x_2 = x_3$, then

$$d(P, R) = |y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3| = d(P, Q) + d(Q, R)$$

Case II $x_1 \neq x_2 \neq x_3$, then

$$d(P, R) = 1 < 1 + 1 = d(P, Q) + d(Q, R)$$

Case III $x_1 \neq x_2 = x_3$, then

$$d(P, Q) = 1, d(Q, R) = |y_2 - y_3|, d(P, R) = |y_1 - y_3|$$

$$\Rightarrow d(P, Q) + d(Q, R) = 1 + |y_2 - y_3| > |y_1 - y_3| = d(P, R)$$

Case IV $x_1 = x_2 \neq x_3$, then

$$d(P, Q) = |y_1 - y_2|, d(Q, R) = 1, d(P, R) = 1$$

$$d(P, Q) + d(Q, R) = |y_1 - y_2| + 1 > 1 = d(P, R)$$

Case V $x_1 \neq x_3$ then

Case a) $x_2 = x_1 = x_3$;

$$d(P, Q) = |y_1 - y_2|, d(Q, R) = 1, d(P, R) = 1$$

$$d(P, R) = 1 < |y_1 - y_2| + 1 = d(P, Q) + d(Q, R)$$

Case b) $x_3 = x_2$;

$$d(P, Q) = 1, d(Q, R) = |y_2 - y_3|, d(P, R) = 1$$

$$d(P, R) = 1 < 1 + |y_2 - y_3| = d(P, Q) + d(Q, R)$$

Thus, d is a metric on $\mathbb{R} \times \mathbb{R}$.

Finally, let U be an open set in metric topology and $P = (x, y) \in \mathbb{R} \times \mathbb{R}$

$\hookrightarrow$ The uniform topology on $\mathbb{R}^J$ is finer than the product topology but coarser than the box topology.
 $(\tau_p \subset \tau_u \subset \tau_b)$ ($\tau_p \subset \tau_u \subset \tau_b$)
 $\S$ All these three topologies are different if J is infinite

$\hookrightarrow \mathbb{R}^\omega$ is metrizable in product topology on $\mathbb{R}^\omega$ is metrizable as:

$$\text{Define } D(x, y) = \sup \left\{ \frac{d(x_i, y_i)}{i} \right\}, \quad x, y \in \mathbb{R}^\omega$$

Then D defines a metric & induces the product topology.

$\hookrightarrow \mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.
 To see this, we defined a metric $d: (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$ as

$$d(xxy, x'xy') = \begin{cases} 0 & \text{if } xxy = x'xy' \\ 1 & \text{if } x \neq x' \text{ or } y \neq y' \\ |y - y'| & \text{if } x = x' \text{ \& } y \neq y' \end{cases}$$

First, we need to verify that d is a metric, for that we've

$$\text{for } d(xxy, xxy) = |y - y| = 0$$

(i) it is easy to see that $d(xxy, x'xy') \geq 0$ $\forall xxy, x'xy' \in \mathbb{R} \times \mathbb{R}$

$$\begin{aligned}
 d(xxy, xxy) = 0 &\Leftrightarrow xxy = x'xy' \\
 \text{Further, } d(xxy, x'xy') = 0 &\Leftrightarrow \begin{cases} x = x' \text{ \& } |y - y'| = 0 \\ \Rightarrow x = x' \text{ \& } y - y' = 0 \\ \Rightarrow x = x' \text{ \& } y = y' \\ \Rightarrow xxy = x'xy' \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } d(xxy, x'xy') &= 1 \text{ (if } x \neq x') = d(x'xy', xxy) \\
 &= |y - y'| \text{ (if } x = x' \text{ \& } y \neq y') = |y - y'| = d(xxy, xxy)
 \end{aligned}$$

Intermediate Value Theorem: let $f: X \rightarrow Y$ be a cont. map where X is a connected space & Y is an ordered set in order topology. If $a, b \in X$ & if $x \in Y$ lying b/w $f(a)$ & $f(b)$, then $\exists c \in X$ s.t. $f(c) = x$.

Definition Given points x, y of space X , a path in X from x to y is a cont. map $f: [a, b] \rightarrow X$ of some closed interval in the real line into X , s.t. $f(a) = x$ & $f(b) = y$. A space X is said to be path connected if every pair of points of X can be joined by a path in X .

$\hookrightarrow$ path-connected $\Rightarrow$ connected (not conversely).

$\hookrightarrow$ Unit ball $B^n = \{x \mid \|x\| \leq 1\}$ in $\mathbb{R}^n$ is path connected.
 $\hookrightarrow$ Euclidean norm.

Connected.

$\hookrightarrow$ Every open ball $B_d(x, \epsilon)$ & every closed ball $\bar{B}_d(x, \epsilon)$ in $\mathbb{R}^n$ is path connected.

$\hookrightarrow$ The punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path connected.

$\hookrightarrow$ Continuous image of a path connected space is path connected.

$\hookrightarrow$ The unit sphere $S^{n-1} := \{x \mid \|x\| = 1\}$ in $\mathbb{R}^n; n \geq 1$ is path connected. $\therefore g: \mathbb{R}^n - \{0\} \xrightarrow{\text{onto}} S^{n-1}$ s.t. $g(x) = \frac{x}{\|x\|}$ is cont.

$\hookrightarrow$ The 'ordered' square I_0^2 is connected but not path connected.

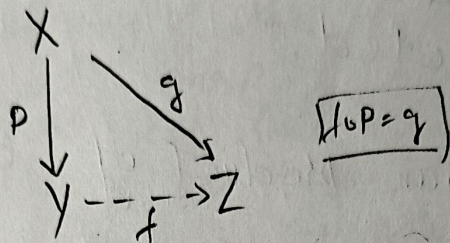
- The union of collection of connected subspaces of X that share a point is connected.
- Let $A \subset X$ be connected. If $A \subset B \subset \bar{A}$ then B is also connected. → closure of a connected space is connected.
- The image of continuous connected space under cont. map is connected.
- A finite Cartesian product of connected spaces is connected.
- $\mathbb{R}^n$ is not connected in box topology, even tho $\mathbb{R}$ is connected.
- $\mathbb{R}^n$ is connected in product topology.
- Arbitrary product of connected spaces is connected in the box topology.
- A space is totally disconnected if its only connected subspaces are one-point sets.

Def A simply ordered set L having more than one element is called a linear continuum 'iff'

- L has least upper bound property
- if $x < y$, $\exists z$ s.t. $x < z < y$.

- if L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L .
- $\mathbb{R}$ is connected & so are rays & intervals in $\mathbb{R}$.

$\hookrightarrow$ Let $p: X \rightarrow Y$ is a quotient map. Let Z be a space & $g: X \rightarrow Z$ is a map s.t. it is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induced a map $f: Y \rightarrow Z$ s.t. $f \circ p = g$.
 f is cont iff g is cont; f is quotient iff g is quotient.



$\hookrightarrow$ A topological group G is a group that is also a topological space satisfying the T_1 -axiom, s.t. the map $f: G \times G \rightarrow G$ s.t. $f(xy) = x \cdot y$ & $g: G \rightarrow G$ s.t. $g(x) = x^{-1}$ are continuous.

Def A separation of a topological space X is a pair of open subsets U, V of X with $U \cap V = \emptyset$ s.t. $X = U \cup V$.
 The space X is said to be connected if there doesn't exist any separation.

$\hookrightarrow$ If a space X is connected, then so is any space homeomorphic to X .

$\hookrightarrow$ A space X is connected iff $\emptyset \neq A \subsetneq X$ are the only subsets of X that are both open & closed.

$\hookrightarrow$ $\mathbb{Q}$ is not connected $\hookrightarrow$ only one-point subsets of $\mathbb{Q}$ are connected subspaces of $\mathbb{Q}$. ($\mathbb{Q}^c$ is not connected)

$\hookrightarrow$ If $X = C \cup D$ is a separation of X & $Y \subseteq X$ is connected then either $Y \subseteq C$ or $Y \subseteq D$.

Quotient Space Let X be a topological space &
 X^* be a partition of X into disjoint subsets whose
 union is X (i.e., $X = \bigcup_{A \in X^*} A$; $A \cap B = \emptyset \neq A, B \in X^*$).
 Let $p: X \rightarrow X^*$ be the surjective map s.t.
 $p(x) = A$ where $x \in A \in X^*$.

Then the quotient topology induced by p , X^* is called
 quotient space of X .

Note Partition $\Leftrightarrow$ equivalence relation (whose eq. classes are elts of partition set)
 Equivalently, A subset U of X^* is a collection of

equivalence classes & $p^{-1}(U) = \bigcup_{A \in U} A = \bigcup_{x \in X} U(x)$.

Thus, U is open ~~iff~~ iff it is a collection of
 equivalence classes whose union is open in X .

Note Restriction of a quotient map to a subspace
 need not be a quotient map.

$\rightarrow$ Let $p: X \rightarrow Y$ be a quotient map; let $A \subset X$
 that is saturated w.r.t p ; let $q: A \rightarrow p(A)$ be the
 map obtained by restricting p .

1) If A is either open or closed, then q is quotient
 map.

2) If p is either open or closed map, then q is quotient
 map.

$\rightarrow$ Composite of two quotient maps is a quotient map.

$\rightarrow$ Cartesian Product of quotient maps need not be quotient

$\rightarrow$ If X is hausdorff, the quotient space X^* need not be as well.

$p: X \rightarrow Y$ is then said to be a quotient map (iff) if p is cont and p maps saturated open subsets of X to open subsets of Y (or saturated closed sets to closed sets).

$\S$ $p: X \xrightarrow{\text{cont}} Y$ is saturated iff $A \subseteq Y$ is closed $\iff p^{-1}(A)$ is closed in X .

$\hookrightarrow f: X \rightarrow Y$ is an open map iff for each open set $U \subseteq X$, $f(U)$ is open in Y .

$\S$ $f: X \rightarrow Y$ is a closed map iff for each closed set $A \subseteq X$, $f(A)$ is closed in Y .

Definition 2 If X is a space & A is a set and if $p: X \rightarrow A$ is a surjective map, then $\exists$ exactly one topology τ on A relative to which p is a quotient map; it is called the quotient topology.

$$\tau = \{ U \subseteq A \mid p^{-1}(U) \text{ is open in } X \}$$

Clearly $p^{-1}(\emptyset) = \emptyset$ & $p^{-1}(A) = X$ are open in X

$\Rightarrow \emptyset, A \in \tau$
Also, the other two conditions follow from it

$$\text{Also, } p^{-1}\left(\bigcup_{\alpha \in I} U_{\alpha}\right) = \bigcup_{\alpha \in I} p^{-1}(U_{\alpha}) \quad \& \quad p^{-1}\left(\bigcap_{\alpha \in I} U_{\alpha}\right) = \bigcap_{\alpha \in I} p^{-1}(U_{\alpha})$$

Let X be a topological space. Then X is compact
iff for every collection $\mathcal{C}$ of closed sets in X
having F.I.P., $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

In finite complement topology on $\mathbb{R}$, every subspace
is compact.

If $C_1 \supset C_2 \supset \dots \supset C_n \supset C_{n+1}$ is a collection of
nested closed set of compact space X , then

$$\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset.$$

Let X be a simply ordered set having least upper
bound property. In order topology, each closed interval
in X is compact.

$\hookrightarrow \mathbb{R}$ with finite complement topology is not Hausdorff
 $\rightarrow$ Every subset of $\mathbb{R}$ is compact in this topology, not only closed sets are finite subsets.

$\hookrightarrow$ The image of a compact space under continuous map is compact.

$\hookrightarrow$ Let $f: X \rightarrow Y$ be a bijective continuous function. If X is compact & Y is Hausdorff, then f is homeomorphism.

$A \subseteq X$ is closed $\Rightarrow f(A)$ is compact ($\because X$ is compact)

$\Rightarrow f(A)$ is compact ($\because f$ is cont) $\Rightarrow f(A)$ is closed ($\because Y$ is Hausdorff)

$\Rightarrow f^{-1}$ is cont $\Rightarrow f$ is homeomorphism.

$\hookrightarrow$ The product of finitely many compact spaces is compact.

The Tube Lemma: Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times V$ about $x_0 \times Y$, where W is a nbd of x_0 in X .
 $\rightarrow x_0 \times Y$ is compact being homeomorphic to Y
 $\rightarrow$ so we can cover $x_0 \times Y$ by finitely many basic elements.

$\hookrightarrow$ The product of finitely many compact spaces is compact.

Def: A collection C of subsets of X is said to have finite intersection property if for every finite subcollection $\{C_1, \dots, C_n\}$ of C , $\bigcap_{i=1}^n C_i \neq \emptyset$.

$\hookrightarrow X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

Let A be an open covering of X , then $\exists U \in A$ s.t. $0 \in U$, then U contains all but finite elements of X . Then, for each $x \in X$ not in U , choose an element of A containing it. This way we obtain a finite subcollection of A that covers X .

$\hookrightarrow$ Any space X containing only finitely many points is necessarily compact.

$\hookrightarrow (0, 1], (0, 2)$ are not compact.

$\hookrightarrow A = \{(\frac{1}{n}, 1] : n \in \mathbb{N}\}$ is not finite subcollection containing covering X .

$\hookrightarrow$ Let Y be a subspace of X . A collection A of subsets of X is said to cover Y if the union of its elements contains Y .

$\hookrightarrow$ Let Y be a subspace of X . Then Y is compact iff for every covering of Y by sets open in X contains a finite subcollection covering Y .

$\hookrightarrow$ Every closed subspace of a compact space is compact.

$\hookrightarrow$ Every compact subspace of a Hausdorff space is closed.

$\hookrightarrow$ If Y is a compact subspace of the Hausdorff space X & x_0 is not in Y , then $\exists$ disjoint open sets U & V of X containing x_0 & Y , respectively.

i.e. if $Y \subseteq X$ is compact & $x_0 \in X - Y$, then $\exists U, V$ with $U \cap V = \emptyset$ where U, V are open in X s.t. $x_0 \in U$ & $Y \subseteq V$.

About Topologist's sine curve

- Connected → not path connected
- locally not locally connected
- Compact

→ A space X is locally connected iff for every open set U of X , each component of U is open in X .

→ A space X is locally path connected iff for every open set U of X , each path component of U is open in X .

→ If X is a topological space, each path component of X is a component of X . If X is path-connected then the components of X are the path components of X .

→ A space X is said to be ^{locally} weakly connected at x if for every nbhd U of x , there is a connected subspace of X contained in U that contains a nbhd of x .

→ A collection A of collection subsets of X is said to cover X , or to be a covering of X , if the union of elements of A equals X . If the elements of A are open as well then it is called an open covering.

→ A space X is said to be **compact** if ~~for~~ every ~~covering~~ open covering A contains a finite subcollection that also covers X .

eg. $\mathbb{R}$ is not compact
→ $A = \{(n, n+2) : n \in \mathbb{Z}\}$ does not contain any open subcoverings.

Common Defn Given X , define an eq. rel. on X by $x \sim y$ if there is a connected subspace of X containing both x & y . The equivalence classes are called components (or "connected components") of X .

$\hookrightarrow$ The components of X are disjoint connected subspaces of X whose union is X , s.t. each subspace of X intersects only one non-empty connected component.

Defn We define another eq. on X by $x \sim y$ if there is a path in X from x to y . The eq. classes are called path-components of X .

$\hookrightarrow$ Each component of a space X is closed in X .
 $\hookrightarrow$ If X has only finitely many components, then each component is open in X as well.

Defn A space X is said to be locally connected at u if for every nbhd U of u , $\exists$ ^{connected} nbhd V of u s.t. $V \subset U$. If X is locally connected at each of its points, it is said to be locally connected.
 $\hookrightarrow$ X is said to be locally path-connected at u if, for every nbhd U of u , $\exists$ ^{path-connected} nbhd V of u s.t. $V \subset U$. If, $\forall u \in X$, X is locally path-connected at u , X is said to be locally path-connected.

e.g., intervals & rays in $\mathbb{R}$
 $\hookrightarrow [-1, 0] \cup (0, 1] \subset \mathbb{R} \rightarrow$ locally connected but not connected
 $\hookrightarrow \mathbb{Q}$ is not connected neither locally connected.
 $\hookrightarrow S = \{x + i \sin(\frac{1}{x}) : 0 < x \leq 1\}$
then $\bar{S} = S \cup \{0\}$ is called topologist's sine curve.

Properties of Cantor set C

- 1) C is totally disconnected.
- 2) C is compact.
- 3) C has no isolated points
- 4) C is uncountable.

5) A_n is union of finitely many disjoint ~~closed~~ intervals of length $\frac{1}{3^n}$ s.t. end of these intervals lie in C .

Def A space X is said to be limit point compact if every infinite subset of X has a limit point.
 Aka Fréchet Compactness or Bolzano-Weierstrass property.

→ Compactness $\Rightarrow$ Limit point compactness (not conversely).

→ Let $Y = [a, b]$ & $T_Y = \{\emptyset, [a, b]$
 Then, $X = \mathbb{Z}_+ \times Y$ is limit point compact but not compact as the open sets $U_n = \{n\} \times Y$ cover X but there doesn't exist any finite subcover.

→ Minimal uncountable ordered set S_ω in order topology is limit point compact but not compact.

Def The space X is said to be sequentially compact if every seq of points of X has a convergent subseq.

→ Let X be a metrizable space. Then following are equivalent:

- (a) X is compact
- (b) X is limit point compact
- (c) X is sequentially compact

→ A space X is said to be **countably compact** if every countable open covering contains a finite subcollection that covers X .

→ Let (X, d) be a metric space. If $f: X \rightarrow X$ is s.t. $d(f(x), f(y)) = d(x, y) \quad \forall x, y \in X$ then f is called an isometry of X .

→ Let (X, d) be a metric space. If $f: X \rightarrow X$ is s.t. $d(f(x), f(y)) < d(x, y) \quad \forall x, y \in X \text{ with } x \neq y$

then f is called a shrinking map.

If there is a $0 < \alpha < 1$ s.t. $d(f(x), f(y)) \leq \alpha d(x, y) \quad \forall x, y \in X$,

then f is called a contraction.

→ A space X is said to be **locally compact at x** if there is some compact subspace C of X that contains a neighbourhood of x .
If X is locally compact at each of its points then X is said to be locally compact.

→ Compactness $\Rightarrow$ locally compactness.

→ $\mathbb{R}$ is locally compact (as, if $x \in \mathbb{R}$, $x \in (a, b)$ for some $a, b \in \mathbb{R}$ & $(a, b) \subset [a, b]$.)

→ $\mathbb{R}^n$ is locally compact

→ $\mathbb{R}^\omega$ is not locally compact.

→ Every simply ordered set X having lub property is locally compact. Given any element for x , it is contained in a closed interval in X .

Properties of Cantor set C

- 1) C is totally disconnected.
- 2) C is compact.
- 3) C has no isolated points.
- 4) C is uncountable.

5) A_n is union of finitely many disjoint closed intervals of length $\frac{1}{3^n}$ s.t. end of these intervals lie in C .

Def A space X is said to be limit point compact if every infinite subset of X has a limit point.
 Also Fréchet Compactness or Bolzano-Weierstrass property.

→ Compactness $\Rightarrow$ Limit point compactness (not conversely).
 → Let $Y = \{a, b\}$ & $\tau_Y = \{\emptyset, \{a, b\}\}$
 Then, $X = \mathbb{Z}_+ \times Y$ is limit point compact but not compact as the open sets $U_n = \{n\} \times Y$ cover X but there doesn't exist any finite subcover.

→ Minimal uncountable ordered set S_ω in order topology is limit point compact but not compact.

Def The space X is said to be sequentially compact if every seq of points of X has a convergent subseq.

→ Let X be a metrizable space. Then following are equivalent:
 (a) X is compact (b) X is limit point compact (c) X is sequentially compact

Def: A function $f: (X, d_x) \rightarrow (Y, d_y)$ is said to be uniformly cont. if given $\epsilon > 0$, $\exists \delta > 0$ st for every pair of points x_0, x_1 of X

$$d_x(x_0, x_1) < \delta \Rightarrow d_y(f(x_0), f(x_1)) < \epsilon.$$

Uniform Cont. Theorem: Let $f: (X, d_x) \rightarrow (Y, d_y)$ be cont. map where (X, d_x) is compact. Then f is uniformly cont.

Def: If X is a space, a point x of X is said to be an isolated point of X if the one-point set $\{x\}$ is open in X .

$\hookrightarrow$ Let X be a non-empty compact Hausdorff space. If X has no isolated point then X is uncountable.

$\hookrightarrow$ Every closed interval in $\mathbb{R}$ is uncountable.

Cantor's Set

Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$.
Let A_2 be the set obtained from A_1 by deleting its middle thirds $(\frac{1}{9}, \frac{2}{9})$ & $(\frac{7}{9}, \frac{8}{9})$.
In general, define A_n as

$$A_n := A_{n-1} - \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

The intersection $C = \bigcap_{n \in \mathbb{N}} A_n$ is called Cantor set. It is a subspace of $[0, 1]$.

- Every closed interval in $\mathbb{R}$ is compact.
- A subspace A of $\mathbb{R}^n$ is compact iff it is closed & is bounded in the Euclidean metric d or S^n metric P . (Heine-Borel Theorem)

$$\Rightarrow \delta \cdot P(x,y) \leq d(x,y) \leq \sqrt{n} P(x,y).$$

- $\mathbb{R}^n$ (under d & P) is Hausdorff
- if a space A of $\mathbb{R}^n$ is compact then it is closed as well.

→ Unit sphere & unit ball are compact in $\mathbb{R}^n$.

Extreme Value Theorem: Let $f: X \rightarrow Y$ be cont, where Y is a real ordered set in the order topology. If X is compact, then $\exists$ points c & d in X s.t. $f(c) \leq f(x) \leq f(d) \forall x \in X$.

→ Let (X,d) be a metric space, $A \subset X$ s.t. $A \neq \emptyset$. If $x \in X$, the distance from x to A is defined as:

$$d(x,A) := \inf \{d(x,a) \mid a \in A\}$$

The function $d(x,A)$ is continuous!!

→ diameter of a bounded subset of metric space (X,d) is $= \sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}$

Lebesgue number lemma: Let A be an open covering of the metric space (X,d) . If X is compact, there is a $\delta > 0$ s.t. for each subset of X having diameter less than δ , $\exists$ an element of A containing it.

→ δ is called Lebesgue number for the covering A .

Countability & Separation

→ A space X is said to have a countable basis at x if there exists a countable collection B of nbds of x s.t. each nbd of x contains atleast one of the elts of B .

→ A space X has countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first countable**.

→ Let X be a topological space.

(a) Let $A \subseteq X$. If there is a seq of pts of A converging to x , then $x \in \bar{A}$. Converse hold if X is first-countable.

(b) Let $f: X \rightarrow Y$ be cont, then $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$ for every cont seq (x_n) . Converse holds if X is first-countable.

Def of a space X has a countable basis for its topology then it is said to satisfy **second-countability axiom** or to be **second-countable**.

→ Second Countability axiom $\Rightarrow$ first countability axiom

→ If X has countable basis, then every any discrete subspace A of X must be countable.

$\hookrightarrow$ Let J be a directed set & $f: J \rightarrow X$ be a net defined as $f(\alpha) = x_\alpha$. If K is a directed set & $g: K \rightarrow J$ is a

(i) $i \leq j \Rightarrow g(i) \leq g(j)$

(ii) $g(K)$ is cofinal in J .

then the composite map $f \circ g: K \rightarrow X$ is called a subnet of (x_α) .

$\hookrightarrow$ If a net converges to x , so does its every subnet.

$\hookrightarrow$ We say that x is an accumulation point (a.c.p.) of the net (x_α) if for each nbhd U of x , the set of those $\alpha \in J$ for which $x_\alpha \in U$ is cofinal in J .

i.e., $\{\alpha \in J \mid x_\alpha \in U\}$ is cofinal in J .

$\hookrightarrow$ The net (x_α) has a a.c.p. x iff some subnet of (x_α) converges to x .

$\hookrightarrow$ X is compact iff every net in X has a convergent subnet.

$\hookrightarrow$ $f: X \rightarrow Y$ is cont. iff $x_\alpha \rightarrow x \Rightarrow f(x_\alpha) \rightarrow f(x)$

→ A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

→ A directed set J is a set with a partial order $\leq$ s.t. for each pair $\alpha, \beta \in J$, $\exists \gamma \in J$ having the property that $\alpha \leq \gamma$ & $\beta \leq \gamma$.

→ A subset K of J is said to be cofinal if, $\forall \alpha \in J$, $\exists \beta \in K$ s.t. $\alpha \leq \beta$.

→ Let X be a topological space. A net in X is a function from a directed set J into X . If $\alpha \in J$, we usually denote $f(\alpha)$ by x_α . We denote the net f itself by $(x_\alpha)_{\alpha \in J}$ or simply as (x_α) .

⇒ Nets are the generalisation of sequences from $\mathbb{N}$ to a general directed set.

→ A net (x_α) is said to converge to the point x of X (i.e., $x_\alpha \rightarrow x$) if for each nbhd U of x , $\exists \alpha \in J$ s.t. $\alpha \leq \beta \Rightarrow x_\beta \in U$.

→ If X is Hausdorff, a net in X converges to at most one point.

→ $x \in \bar{A}$ iff there is a net of points of A converging to x .

Let X be a space. The X is locally compact. (10)
Hausdorff iff $\exists$ a space Y satisfying:

(1) X is a subspace of Y .

(2) The set $Y - X$ consists of single point.

(3) Y is compact Hausdorff space.

If Y & Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity on X .

Def of Y is a compact Hausdorff space & X is a subspace of Y s.t. $\bar{X} = Y$, then Y is said to be compactification of X .

If, $Y - X = \{p\}$, then Y is called one-point compactification of X .
in addition

→ One-point compactification of:

& $\mathbb{R}$ is homeomorphic to with the circle.

& $\mathbb{R}^2$ is " to the sphere S^2 .

So if $\mathbb{R}^2$ is looked as the space $\mathbb{C}$, then $\mathbb{C} \cup \{\infty\}$ is called the Riemann sphere or the extended complex plane.

→ Hausdorff space X is locally compact iff given $x \in X$,
& a nbhd U of x , there is a nbhd V of x s.t.
 $\bar{V}$ is compact & $\bar{V} \subset U$.

→ Let X be locally compact Hausdorff: let A be a subspace of X . If A is closed or open in X , then A is locally compact.

→ A space X is said to be **countably compact** if every countable open covering contains a finite subcollection that covers X .

→ Let (X, d) be a metric space. if $f: X \rightarrow X$ is s.t. $d(f(x), f(y)) = d(x, y) \quad \forall x, y \in X$ then f is called an isometry of X .

→ Let (X, d) be a metric space. if $f: X \rightarrow X$ is s.t. $d(f(x), f(y)) < d(x, y) \quad \forall x, y \in X \text{ with } x \neq y$ then f is called a shrinking map.

if there is a α s.t. $d(f(x), f(y)) \leq \alpha d(x, y) \quad \forall x, y \in X$, then f is called a contraction.

→ A space X is said to be **locally compact** if there is some compact subspace C of X that contains a neighbourhood of x .

if X is locally compact at each of its points then X is said to be locally compact.

→ Compactness $\Rightarrow$ locally compactness.

→ $\mathbb{R}$ is locally compact (as, if $x \in \mathbb{R}$, $x \in (a, b)$ for some $a, b \in \mathbb{R}$ & $(a, b) \subset [a, b]$.)

→ $\mathbb{R}^n$ is locally compact

→ $\mathbb{R}^\omega$ is not locally compact.

→ Every simply ordered set X having lub property is locally compact. Given any element for X , it is contained in a closed interval in X .

The Urysohn Lemma

Statement: Let X be a normal space; let A & B be disjoint closed ~~subspaces~~ subsets of X . Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f: X \rightarrow [a, b]$

such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

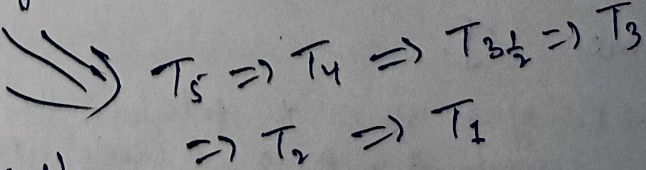
Definition: If A & B are two subsets of the topological space X , and if there is a continuous function $f: X \rightarrow [0, 1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be separated by a continuous function.

Definition: A space X is completely regular if one-point sets are closed in X and if for each point x_0 & each closed set A not containing x_0 , there is a continuous function $f: X \rightarrow [0, 1]$ s.t. $f(x_0) = 1$ and $f(A) = \{0\}$.

→ By Urysohn lemma, a normal space is completely regular.
 → Completely regular $\Rightarrow$ regular. $\left(\begin{matrix} f^{-1}([0, \frac{1}{2})) & \& f^{-1}([\frac{1}{2}, 1]) \\ \text{are two open sets ending} & & \\ A & \& x_0 \text{ resp.} \end{matrix} \right)$

Separation Axioms

- $T_1 \rightarrow$ Fréchet space $\rightarrow$ For every pair $x, y \in X$ $\exists$ open nbds U of x & V of y s.t. $y \in U$ & $x \notin V$. Or, equivalently, every singleton set is closed.
- $T_2 \rightarrow$ Hausdorff
- $T_3 \rightarrow$ regular
- $T_4 \rightarrow$ normal
- $T_5 \rightarrow$ Completely normal (or every subspace normal)
- $T_{3\frac{1}{2}} / T_{3\frac{1}{2}} \rightarrow$ Completely regular
- Subspace of a completely regular & regular space is again completely regular. $\Rightarrow$ Product of completely regular spaces is completely regular.



Summary: Regular space (with countable basis), ~~normal~~ metrizable, Compact Hausdorff, ~~well-ordered set~~ order topology $\Rightarrow$ normal

- Product of normal spaces may not be normal.
- Uncountable product of normal spaces is not normal.
- Regular space need not be normal.
- Subspace of a normal space need not be normal.

$\hookrightarrow S_\omega \vee \bar{S}_\omega$ is not normal (ie. Product of normal spaces need not be normal).

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 $\hookrightarrow$ both with order topology $\Rightarrow$ both are normal.

$\hookrightarrow$ Product of regular spaces

$\hookrightarrow \bar{S}_\omega \times \bar{S}_\omega$ is compact Hausdorff, & therefore, normal.

let X be a topological space. let one-pt sets $\{x\}$ be closed.

(a) X is regular $\Leftrightarrow$ given a point x of X & a nbd U of x , $\exists$ nbd V of x s.t. $\overline{V} \subset U$.

(b) X is normal $\Leftrightarrow$ given a closed set A & an open set U containing A $\exists$ an open set V containing A s.t. $\overline{V} \subset U$.

$\hookrightarrow$ A subspace of Hausdorff space is Hausdorff.
§ Product of Hausdorff spaces is Hausdorff.

$\hookrightarrow$ A subspace of regular space is regular.
§ Product of regular spaces is regular.

$\hookrightarrow \mathbb{R}_\kappa$ is Hausdorff but not regular.

$\hookrightarrow \mathbb{R}_\kappa$ is normal.

$\hookrightarrow$ The Sorgenfrey plane $\mathbb{R}_\ell^2$ is not normal.

§ $\mathbb{R}_\ell$ is regular (in fact, normal)

$\Rightarrow \mathbb{R}_\ell^2$ is regular ($\Rightarrow$ regular $\nRightarrow$ normal)

§ Product of two normal spaces need not be normal.

$\hookrightarrow$ Every regular space with countable basis is normal.

$\hookrightarrow$ Every metrizable space is normal.

$\hookrightarrow$ Every compact Hausdorff space is normal.

$\hookrightarrow$ Every well-ordered set is normal in order topology.

In fact, every order topology is normal.

$\hookrightarrow$ If J is uncountable, the product space $\mathbb{R}^J$ is not normal.

→ The product of two Lindelöf spaces need not be Lindelöf. (as $\mathbb{R}_l$ is Lindelöf but $\mathbb{R}_l \times \mathbb{R}_l$ is not)

→ $\mathbb{R}_l \times \mathbb{R}_l = \mathbb{R}_l^2 \rightarrow$ Sorgenfrey Plane.

→ A subspace of a Lindelöf space need not be Lindelöf.

The ordered sq. I_0^2 is compact, therefore Lindelöf.
However $A = I \times (0, 1)$ is not Lindelöf.

→ Suppose that one-point sets are closed in X .

Regular: if for each pair of consisting of a pt x & a closed set B disjoint from x , $\exists$ ~~open~~ disjoint open set containing x & B respectively.

Normal: if for each pair A, B of disjoint closed sets, $\exists$ disjoint open sets containing A & B resp.

Fréchet Space → One-point sets are closed → T_1

Hausdorff → Two points ^{can be separated by disjoint open sets} → T_2

Regular → One point & one closed set → T_3

Normal → Two closed sets (disjoint) → T_4

Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff. (Assuming one-pt sets are closed)

Fréchet Space: For every pair x, y with $x \neq y$, $\exists$ open set containing x but not y & likewise one containing y but not x .
→ Every finite set in T_2 -space is closed.

- A subspace of a first-countable space is ^{first-}countable.
- Countable product of first-countable spaces is first-countable.
- A subspace of a second-countable space is second-countable.
- Countable product of second-countable spaces is second-countable.

~~Def~~ A subset A of X is said to be **dense** in X if $\bar{A} = X$.

- A space X is a second-countable space. Then,
- every open covering of X contains a countable open covering.
 - There exists a countable dense subset of X .

→ A space for which the above two conditions are equivalent to second-countability is called **metrizable**.

→ A space for which every open covering contains a countable subcovering is called a **Lindelöf** space.

→ A space having a countable dense subset is called (often) to be **separable**. (not confusing with the separability we've already done)

→ $\mathbb{R}$ satisfies all the countability axioms but the second.

↳ Lindelöf & First countable $\exists A \subset \mathbb{R}$ s.t. $\bar{A} = \mathbb{R}$ (in fact, $A = \mathbb{Q}$).

If Y is called compactification induced by the immediate h .

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Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded cont. map $f: X \rightarrow \mathbb{R}$ extends uniquely to a cont. map of Y into $\mathbb{R}$.

Lemma Let $f: A \rightarrow Z$ $A \subset X$; let $f: A \rightarrow Z$ be a cont. map of A into Hausdorff space Z . There is at most one extension of f to a cont. function $g: \bar{A} \rightarrow Z$.

Let X be a completely regular space; let Y be a compactification of X satisfying the extension property of 30.2. Given any cont. map $f: X \rightarrow C$ into a compact Hausdorff space C , the map f extends uniquely to a cont. map $g: Y \rightarrow C$.

Let X be a CRS. If Y_1 & Y_2 are two compactifications of X satisfying 30.2. then Y_1 & Y_2 are equivalent.

Definition For each completely regular space X , let us choose, once and for all, a compactification of X satisfying the condition of Th 30.2. We'll denote this compactification of X by $\beta(X)$ & call it Stone-Čech compactification of X . It is characterized by the fact that any cont. map $f: X \rightarrow C$ of X into a compact Hausdorff space C extends uniquely to a cont. map $g: \beta(X) \rightarrow C$.

Note C ; compact Hausdorff $\Rightarrow$ completely regular $\Rightarrow C$ can be imbedded into $[0, 1]^J$ for some J .

$\hookrightarrow$ let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the F.F.P. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the F.F.P., and no collection of subsets of X that properly contains $\mathcal{D}$ has this property. Such a $\mathcal{D}$ is called maximal w.r.t F.F.P.

$\hookrightarrow$ let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal w.r.t F.F.P. Then,

- Any finite intersection of sets of $\mathcal{D}$ is an element of $\mathcal{D}$.
- If A is a subset of X that intersects every set of $\mathcal{D}$, then $A \in \mathcal{D}$.

Tychonoff Theorem An arbitrary product of compact spaces is compact in the product topology.

$\hookrightarrow$ A collection $\mathcal{A}$ of subsets of X has the countable intersection property if every countable intersection of sets of $\mathcal{A}$ is nonempty.

$\hookrightarrow$ X is a Lindelöf space iff for every collection $\mathcal{A}$ of subsets of X having C.I.P., $\bigcap_{A \in \mathcal{A}} A$ is nonempty.

Def: A compactification of a space X is a compact Hausdorff space Y containing X as a subspace st $\bar{X} = Y$.
 Two compactifications Y_1 & Y_2 of X are said to be equivalent if there is a homeomorphism $h: Y_1 \rightarrow Y_2$ st $h|_X = \text{id}_X$.

$\hookrightarrow$ let X be a space; suppose that $h: X \rightarrow Z$ is an embedding of X in the compact Hausdorff space Z . Then, for $\exists$ a unique compactification Y of X such that h equals $h \circ i$ on X . Here $i: X \rightarrow Y$ is the inclusion map. The compactification Y is unique up to equivalence.

Manifolds

Definition An m -manifold is a Hausdorff space X with a countable basis such that each point x of X has a neighbourhood that is homeomorphic to with an open subset of $\mathbb{R}^m$.

A 1-manifold is often called a curve, and a 2-manifold is called a surface.

$\hookrightarrow$ If $\phi: X \rightarrow \mathbb{R}$, then the support of ϕ is defined to be the closure of the set $\phi^{-1}(\mathbb{R} - \{0\})$. Thus if x lies outside the support of ϕ , there is some neighbourhood of x on which ϕ vanishes.

Definition ^{by compactness} Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of the space X . An indexed family of continuous functions _{Compact?}

$$\phi_i: X \rightarrow [0, 1] \quad \text{for } i = 1, 2, \dots, n$$

is said to be a partition of unity dominated by $\{U_i\}$ if:

- (1) $\text{support of } \phi_i \subset U_i$ for each i (i.e., $\phi_i^{-1}(\mathbb{R} - \{0\}) \subset U_i$)
- (2) $\sum_{i=1}^n \phi_i(x) = 1$ for each x .

Note: General def. is for arbitrary open covering $\{U_i\}_{i \in I}$.

Existence: Let $\{U_1, U_2, \dots, U_n\}$ be a finite open covering of the normal space X . Then there exists a partition of unity dominated by $\{U_i\}$.

Theorem If X is a compact m -manifold, then X can be imbedded in $\mathbb{R}^N$ for some positive integer N .

Tietze Extension Theorem

Let X be a normal space; let A be a closed ~~sub~~ subspace of X .

(a) Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a cont. map of all of X into $[a, b]$

(b) Any cont. map of A into $\mathbb{R}$ may be extended to a cont. map of all of X into $\mathbb{R}$.

$\mathbb{R}^2$ & $S_n \times S_n$ are completely regular, but not normal. For they are products of spaces that are completely regular (in fact, normal).

Urysohn's metrization theorem Every regular space X with a countable basis is metrizable.

Embedding Theorem: Let X be a space in which one-point sets are closed. Suppose that $\{U_\alpha\}_{\alpha \in J}$ is an indexed family of cont. function $f_\alpha: X \rightarrow \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each nbhd U of x_0 , there is an index α s.t. $f_\alpha(x_0) > 0$ and f_α vanishes outside U (i.e., $f_\alpha(X - U) = \{0\}$). Then the function $F: X \rightarrow \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0, 1]$ for each α , then F imbeds X in $[0, 1]^J$.

A family of function that satisfies the hypothesis of this theorem is said to separate points from closed sets in X .

Theorem (Corollary): A space X is completely regular iff it is homeomorphic to a subspace of $[0, 1]^J$ for some J .

Note: If a space X satisfies the condition of imbedding theorem & $x \notin A$ & A is a closed set disjoint from x then $\exists \alpha \in J$ s.t. $f_\alpha(x) > 0$ & $f_\alpha(A) = \{0\}$ (as $X - A$ is open containing x & f_α vanishes outside $X - A$ (i.e., A)). which is also equivalent to def. of completely regular (hence the corollary)