Algebraic Topology: Study Notes

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Note

These notes present a gradual introduction to algebraic topology, beginning with the fundamental concepts of homotopy and the fundamental group before progressing to more advanced topics as the theory develops. The exposition emphasizes precise definitions, rigorous proofs, motivating examples, and geometric intuition, with the aim of building a coherent understanding of the subject. This is a living document that will be continually revised, expanded, and refined as my study of algebraic topology progresses, with new material, examples, exercises, and references incorporated over time.

Contents

Chapter 1

Fundamental Group

The purpose of this chapter is to introduce the fundamental group, one of the central invariants in algebraic topology. We begin by studying paths and homotopies, which naturally lead to the notion of path homotopy. These ideas provide the foundation for defining the fundamental group and exploring its basic properties.

1.1 Homotopy and Path Homotopy

Definition 1.1.1. Let $f, f' : X \to Y$ be continuous maps.
We say that f **is homotopic to** f' if there exists a continuous map

$$F : X \times I \longrightarrow Y, \qquad I = [0, 1],$$

such that

$$F(x, 0) = f(x), \qquad F(x, 1) = f'(x),$$

for every $x \in X$.

- F is called a **homotopy** between f and f'.
- We write
$$f \simeq f'.$$
- If f' is a constant map, then f is called **null-homotopic**.

Geometric Interpretation Think of the parameter $t \in [0, 1]$ as representing time.

- At $t = 0$, the map is f.
- At $t = 1$, the map is f'.
- As t varies continuously from 0 to 1, the homotopy F continuously deforms f into f'.

Definition 1.1.2. A **path** in X is a continuous map

$$f : [0, 1] \longrightarrow X$$

such that

$$f(0) = x_0, \qquad f(1) = x_1.$$

- x_0 is called the **initial point**.
- x_1 is called the **final point**.

Definition 1.1.3. Let $f, f' : [0,1] \to X$ be two paths having the same initial and final points. They are said to be **path homotopic** if there exists a continuous map

$$F : I \times I \longrightarrow X$$

such that

$$F(s,0) = f(s), \qquad F(s,1) = f'(s),$$

and

$$F(0,t) = x_0, \qquad F(1,t) = x_1,$$

for every $s, t \in I$.

- F is called a **path homotopy** between f and f'.
- We write
$$f \simeq_p f'.$$

Remark 1.1.4. Homotopy ($\simeq$) is a continuous deformation of one map into another. Path Homotopy ($\simeq_p$) is a continuous deformation of one path into another while keeping the initial and final points fixed throughout the deformation.

Equivalently,

$$\begin{aligned} \text{Homotopy:} \qquad & F(s,0) = f(s), \quad F(s,1) = f'(s). \\ \text{Path Homotopy:} \qquad & F(s,0) = f(s), \quad F(s,1) = f'(s), \\ & F(0,t) = x_0, \quad F(1,t) = x_1. \end{aligned}$$

Thus, in a path homotopy the endpoints remain fixed throughout the deformation.

Remark 1.1.5 (Pasting Lemma)**.** Let X and Y be topological spaces, and suppose

$$X = A \cup B,$$

where A and B are closed subsets of X.
If

$$f : A \to Y \quad \text{and} \quad g : B \to Y$$

are continuous maps such that

$$f(x) = g(x) \qquad \forall \qquad x \in A \cap B,$$

then the map

$$h : X \to Y, \qquad h(x) := \begin{cases} f(x), & x \in A, \\ g(x), & x \in B, \end{cases}$$

is well-defined and continuous.

Lemma 1.1.1. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*
If f is a path, we denote its path homotopy equivalence class by

$$[f].$$

Proof. We verify the three properties of an equivalence relation.

Reflexivity. Given a map f, define

$$F(x,t) = f(x).$$

Then

$$F(x,0) = f(x) = F(x,1),$$

so $f \simeq f$. If f is a path, then F is also a path homotopy, hence

$$f \simeq_p f.$$

Symmetry. Suppose

$$f \simeq f'$$

via a homotopy F. Define

$$G(x,t) = F(x, 1-t).$$

Then

$$G(x,0) = f'(x), \qquad G(x,1) = f(x),$$

so

$$f' \simeq f.$$

If F is a path homotopy, then G is also a path homotopy, giving

$$f' \simeq_p f.$$

Transitivity. Suppose

$$f \simeq f' \quad \text{and} \quad f' \simeq f''.$$

Let F be a homotopy from f to f' and F' a homotopy from f' to f''.
Define

$$G(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2}, \\ F'(x,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Since

$$F(x,1) = f'(x) = F'(x,0),$$

the map G is well defined. Since G is continous on the two closed sets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$, it is continous on all of $X \times [0,1]$, by pasting lemma.
Moreover,

$$G(x,0) = f(x), \qquad G(x,1) = f''(x),$$

so G is a homotopy between f and f''. Hence

$$f \simeq f''.$$

If F and F' are path homotopies, then G also fixes the endpoints throughout the deformation. Therefore,

$$f \simeq_p f''.$$

Thus both $\simeq$ and $\simeq_p$ are equivalence relations. □

Example 1.1.6. Let f and g be any two maps of a space X in $\mathbb{R}^2$. Then f and g are homotopic; the map

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between them called as a **straight-line homotopy** because it moves the point $f(x)$ to the point $g(x)$ along the straight line joining them.

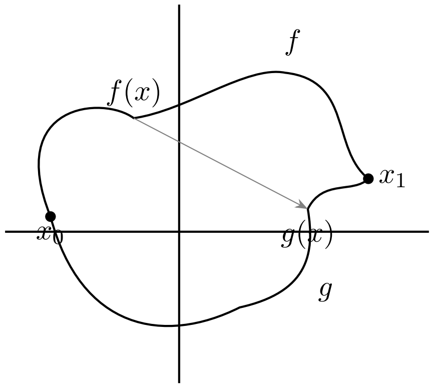


Figure 1.1: Straight-line homotopy between two paths.

Example 1.1.7. Let A be any convex subspace of $\mathbb{R}^n$. Then any two paths f, g in A from x_0 to x_1 are path homotopic in A by the straight-line homotopy F.

Example 1.1.8. Let

$$X = \mathbb{R}, \qquad f(x) = x, \qquad g(x) = 2x.$$

A homotopy between f and g is given by the straight-line homotopy

$$H(x,t) = (1-t)f(x) + tg(x).$$

Substituting f and g, we obtain

$$H(x,t) = (1-t)x + t(2x) = (1+t)x.$$

Clearly,

$$H(x,0) = x = f(x),$$

and

$$H(x,1) = 2x = g(x).$$

Thus H continuously deforms the map f into the map g.

1.1.1 Product of Paths

Definition 1.1.9. Let

$$f : I \to X$$

be a path from x_0 to x_1, and let

$$g : I \to X$$

be a path from x_1 to x_2.
The *product* (or concatenation) of f and g, denoted by

$$f * g,$$

is the path $h : I \to X$ defined by

$$h(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2}, \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since

$$f(1) = x_1 = g(0),$$

the map h is well-defined and continuous (by the Pasting Lemma). Thus h is a path from x_0 to x_2.

Interpretation. The path $h = f * g$ first traverses the path f (during the first half of the interval $[0, 1]$) and then traverses the path g (during the second half).

The concatenation of paths induces a well-defined operation on path-homotopy classes:

$$[f] * [g] = [f * g].$$

That is, the product of the homotopy classes is the homotopy class of the concatenated path. To see this, let

$$F : f \simeq_p f'$$

and

$$G : g \simeq_p g'$$

be path homotopies. Define

$$H(s,t) = \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2}, \\ G(2s-1,t), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since

$$F(1,t) = x_1 = G(0,t)$$

for all $t \in I$, H is well-defined and continuous by the Pasting Lemma. Hence,

$$f * g \simeq_p f' * g'.$$

Therefore, the operation $*$ is well-defined on path-homotopy classes.

Remark. The operation $*$ satisfies properties similar to the group axioms. However, $[f] * [g]$ is defined only when

$$f(1) = g(0),$$

so the collection of path-homotopy classes forms a *groupoid* rather than a group.

1.1.2 Properties of the Product Operation

Before analysing the prperties of the operation '$*$', we note the following preliminary facts that will be used in the arguments of proofs those properties:

1. If $k : X \to Y$ is continuous and

   $$F : I \times I \longrightarrow X$$

   is a path homotopy between the paths f and f', then

   $$k \circ F : I \times I \longrightarrow Y$$

   is a path homotopy between the paths $k \circ f$ and $k \circ f'$.

2. If $k : X \to Y$ is continuous and $f(1) = g(0)$, then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

   This follows immediately from the definition of the product operation.

3. If $[a, b]$ and $[c, d]$ are two intervals in $\mathbb{R}$, there is a unique map

$$p : [a, b] \longrightarrow [c, d]$$

   of the form

$$p(x) = mx + k$$

   that carries a to c and b to d, known as the positive linear map of $[a, b]$ to $[c, d]$ because its graph is a straight line with a postive slope. In fact, inverse of such a map and composite of two such maps are again maps of same kind.

   With this terminology, the product $f * g$ can be described as follows: On $[0, \frac{1}{2}]$, it equals the positive linear map of $[0, \frac{1}{2}]$ to $[0, 1]$, followed by f; and on $[\frac{1}{2}, 1]$, it equals the postive linear map of $[\frac{1}{2}, 1]$ to $[0, 1]$, followed by g. In other words, if $\phi : [0, \frac{1}{2}] \longrightarrow [0, 1]$ is a positive linear map, then $f * g$ on $[0, \frac{1}{2}]$ equals the composition $f \circ \phi$; is traversing the same path f but at twice the speed. A similar argument follows for $[\frac{1}{2}, 1]$.

We now verify the three basic properties of the operation $*$.

Associativity.

If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$, and

$$[f] * ([g] * [h]) = ([f] * [g]) * [h].$$

Proof. Given paths f, g, and h in X, the products $f * (g * g)$ and $(f * g) * h$ are defined precisely when $f(1) = g(0)$ and $g(1) = h(0)$. Now let

$$0 < a < b < 1.$$

Define a path

$$k_{a,b} : I \to X \text{ as follows:}$$

$K_{a,b}$ equals the positive linear maps of $[0, a]$, $[a, b]$, and $[b, 1]$ to I followed by the paths f, g, and h successively on the intervals $[0, a]$, $[a, b]$, and $[b, 1]$, respectively.
The path $k_{a,b}$ depends on the choice of a and b, but its path-homotopy class does not. Indeed, if

$$0 < c < d < 1,$$

define a piecewise linear map

$$p : I \to I$$

which maps

$$[0, a] \mapsto [0, c], \qquad [a, b] \mapsto [c, d], \qquad [b, 1] \mapsto [d, 1]$$

linearly. When restricted to $[0, a]$, $[a, b]$, and $[b, 1]$, respectively, it equals the positive linear map of these intervals onto the intervals $[0, c]$, $[c, d]$, and $[d, 1]$, respectively. Consequently,

$$k_{c,d} \circ p = k_{a,b}$$

.

Since p is a path in I from 0 to 1, it is path homotopic to the identity map $i : I \to I$. Hence, there is a path homotopy P in I between p and i. By the first preliminary fact,

$$k_{c,d} \circ p \simeq_p k_{c,d} \circ i.$$

and therefore

$$k_{a,b} \simeq_p k_{c,d}.$$

Now choose

$$a = \frac{1}{2}, \qquad b = \frac{3}{4}.$$

Then

$$k_{a,b} = f * (g * h).$$

Similarly, choosing

$$c = \frac{1}{4}, \qquad d = \frac{1}{2},$$

gives

$$k_{c,d} = (f * g) * h.$$

Hence these two products are path homotopic. □

The proof above establishes a stronger result: the path-homotopy class of a concatenation is independent of how the interval is subdivided. This is recorded in the following theorem.

Theorem 1.1.2. *Let f be a path in X, and let*

$$0 = a_0 < a_1 < \cdots < a_n = 1.$$

For each $i = 1, \ldots, n$, let $f_i : I \to X$ be the path that equals the postive linear map of I onto the subinterval $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * [f_2] * \cdots * [f_n].$$

Identity.

Let $e_x : I \to X$ denote the constant path at x.
If f is a path from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f], \qquad [e_{x_0}] * [f] = [f].$$

Proof. Since the interval I is convex, the identity map $i : I \to I$ is path homotopic to each of the maps $e_0 * i$ and $i * e_1$. Composing these homotopies with f and using the first two preliminary facts gives

$$f \circ i = f \simeq_p f \circ (i * e_1) = (f \circ i) * (f \circ e_1) = f * e_{x_1},$$

and

$$f \circ i = f \simeq_p f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f.$$

Hence the corresponding path-homotopy classes are equal. □

Inverse.

Let f be a path from x_0 to x_1. Define its reverse path by

$$\bar{f}(s) = f(1-s).$$

Then

$$[f] * [\bar{f}] = [e_{x_0}], \qquad [\bar{f}] * [f] = [e_{x_1}].$$

Proof. Note that the inverse of i is $\bar{i}(s) = 1 - s$. Then, $i * \bar{i}$ is a path in I with starting and ending point 0. Because I is convex, there is a path homotopy between e_0 and $i * \bar{i}$. Composing this homotopy with f and using first two preliminary facts to get

$$f \circ e_0 = e_{x_0} \simeq_p f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \bar{f}$$

Similarly, using the fact that $\bar{i} * i$ is path homotopic in I to e_1, shows that

$$\bar{f} * f \simeq_p e_{x_1}.$$

Therefore,

$$[f] * [\bar{f}] = [e_{x_0}], \qquad [\bar{f}] * [f] = [e_{x_1}].$$

□

In conclusion, if we denote by $P(X, x, y)$ the set of all the paths in X from x to y then the qoutient set $\frac{P(X,x,y)}{\simeq_p}$(the set of all path homotopy classes) does not form a group under the operation '$*$' defined as

$$[f] * [f'] = [f * f'].$$

But under some restrictions, we can remove this imperfection from it by cosidering only those paths which are loops; it shall be discussed in next section.

Remark 1.1.10. Let $h, h' : X \longrightarrow Y$ are homotopic and $k, k' : Y \longrightarrow Z$ are homotopic; let H, K be the respective homotopies. Define $F : X \times I \longrightarrow Z$ as

$$H(x, t) = G(H(x, t), t)$$

. Then F is well-defined and continous. Also,

$$F(x, 0) = G(H(x, 0), 0) = G(h(x), 0) = k(h(x)) = (k \circ h)(x) \text{ and}$$

$$F(x, 1) = G(H(x, 1), 1) = G(h'(x), 1) = k'(h'(x)) = (k' \circ h')(x)$$

Consequently, $k \circ h \simeq k' \circ h'$.

1.2 The Fundamental Group

In the previous section, we saw that the collection of all path-homotopy classes in a space X does not form a group under the product operation $*$, since the product is defined only when the endpoint of the first path coincides with the initial point of the second.
To overcome this difficulty, we fix a point $x_0 \in X$, called the *base point*, and consider only those paths that both begin and end at x_0. The set of path-homotopy classes of such loops is closed under the product operation and forms a group, called the *fundamental group* of X.
The fundamental group is one of the most important algebraic invariants in topology. It captures essential information about the shape of a space and remains unchanged under homeomorphisms. Throughout this section, we shall study its construction and establish some of its fundamental properties.

1.2.1 Fundamental Group

Notation 1.2.1. We denote the set of all the loops, the paths having same starting and end points, in X at x by $\Omega(X, x)$. That is,

$$\Omega(X, x) = P(X, x, x)$$

Definition 1.2.2. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the **fundamental group** of X relative to the **base point** x_0. It is denoted by $\pi_1(X, x_0)$. That is,

$$\pi_1(X, x_0) = \frac{\Omega(X, x_0)}{\simeq_p}.$$

Theorem 1.2.1. *$(\pi_1(X, x_0), *)$ is a group.*

Proof. It follows immediately from the properties established in the previous section that the operation $*$, when restricted to path-homotopy classes of loops based at x_0, satisfies the group axioms. Indeed, the product of two loops based at x_0 is again a loop based at x_0, the constant loop e_{x_0} serves as the identity element, and the reverse of a loop represents its inverse. Thus, $\pi_1(X, x_0)$ forms a group under the operation $*$. $\square$

The group $\pi_1(X, x_0)$ is called the *fundamental group* of X based at x_0. It is also referred to as the *first homotopy group* of X, since it is the first member of the family of homotopy groups

$$\pi_n(X, x_0), \qquad n \geq 1.$$

Example 1.2.3. Let $\mathbb{R}^n$ denote Euclidean n-space. Then

$$\pi_1(\mathbb{R}^n, x_0)$$

is the trivial group. Indeed, if f is a loop in $\mathbb{R}^n$ based at x_0, then the straight-line homotopy between f and the constant loop at x_0 shows that f is null-homotopic. More generally, if X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is trivial. In particular, the unit ball

$$B^n = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$$

has trivial fundamental group.

Example 1.2.4. If $X = x$ is one point space, then only path in X is constant one. Hence, $\pi_1(X, x_0) = [e_x]$ is the trivial group.

Definition 1.2.5. Let α be a path in X from x_0 to x_1. Define the map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha],$$

where $\bar{\alpha}$ denotes the reverse of α.
The map $\hat{\alpha}$ is well-defined since the operation $*$ is well-defined on path-homotopy classes. Indeed, if f is a loop based at x_0, then

$$\bar{\alpha} * (f * \alpha)$$

is a loop based at x_1. Moreover, $\hat{\alpha}$ depends only on the path-homotopy class of α.

Theorem 1.2.2. *The map*

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

is a group isomorphism.

Proof. To show that $\hat{\alpha}$ is a homomorphism, let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * ([f] * [g]) * [\alpha] \\ &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g]),\end{aligned}$$

where associativity and the identities

$$[\alpha] * [\bar{\alpha}] = [e_{x_0}], \qquad [\bar{\alpha}] * [\alpha] = [e_{x_1}]$$

have been used.
To prove that $\hat{\alpha}$ is an isomorphism, let $\beta = \bar{\alpha}$. Then

$$\hat{\beta} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$

is defined by

$$\hat{\beta}([h]) = [\alpha] * [h] * [\bar{\alpha}].$$

For every $[h] \in \pi_1(X, x_1)$,

$$\begin{aligned}\hat{\alpha}(\hat{\beta}([h])) &= [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] \\ &= [h].\end{aligned}$$

Similarly, for every $[f] \in \pi_1(X, x_0)$,

$$\hat{\beta}(\hat{\alpha}([f])) = [f].$$

Hence,

$$\hat{\beta} = \hat{\alpha}^{-1},$$

and therefore $\hat{\alpha}$ is a group isomorphism. □

Corollary 1.2.3. *If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Proposition 1.2.4. *$\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$ if X and Y are path connected.*

Proof. A basic property of the product topology is that a map $f : Z \longrightarrow X \times Y$ is continuous iff the maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ defined by $f(z) = (g(z), h(z))$ are both continuous. Hence a loop f in $X \times Y$ based at (x_0, y_0) is equivalent to a pair of loops g in X and h in Y based at x_0 and y_0 respectively. Similarly, a homotopy F of a loop in $X \times Y$ is equivalent to a pair of homotopies G and H of the corresponding loops in X and Y. Thus we obtain a bijection $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$, $[f] \mapsto ([g], [h])$. This is obviously a group homomorphism, and hence an isomorphism. □

1.2.2 Functoriality

Let

$$h : (X, x_0) \longrightarrow (Y, y_0)$$

be a continuous map.
If f is a loop in X based at x_0, then the composite

$$h \circ f : I \longrightarrow Y$$

is a loop in Y based at y_0. Hence h induces a map

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0),$$

defined by

$$[f] \longmapsto [h \circ f].$$

The map h_* is called the *homomorphism induced by* h, relative to the base point x_0.
The map h_* is well-defined. Indeed, if

$$F : f \simeq_p g,$$

then

$$h \circ F : I \times I \longrightarrow Y$$

is a path homotopy between $h \circ f$ and $h \circ g$. Hence

$$[f] = [g] \quad \Longrightarrow \quad [h \circ f] = [h \circ g].$$

Moreover, h_* is a group homomorphism since

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

Using the induced homomorphism, we obtain the following properties.

Theorem 1.2.5 (Functorial properties)**.** *Let* $(X, x_0) \xrightarrow{h} (Y, y_0) \xrightarrow{k} (Z, z_0)$ *be continuous maps. Then:*

1. $(k \circ h)_* = k_* \circ h_*$.
2. *If* $i : (X, x_0) \longrightarrow (X, x_0)$ *is the identity map, then* $i_* = \mathrm{id}_{\pi_1(X,x_0)}$.

Proof. For any $[f] \in \pi_1(X, x_0)$,

$$\begin{aligned}
(k \circ h)_*([f]) &= [(k \circ h) \circ f] \\
&= [k \circ (h \circ f)] \\
&= k_*([h \circ f]) \\
&= k_*(h_*([f])) \\
&= k_* \circ h_*([f])).
\end{aligned}$$

Hence

$$(k \circ h)_* = k_* \circ h_*.$$

Similarly, $i_*([f]) = [i \circ f] = [f]$, so $i_* = \mathrm{id}_{\pi_1(X,x_0)}$. □

Corollary 1.2.6. *The fundamental group* $\pi_1(X, x_0)$ *is a topological invariant. That is, if*

$$h : (X, x_0) \longrightarrow (Y, y_0)$$

is a homeomorphism, then

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is a group isomorphism.

Proof. Let

$$k = h^{-1}.$$

Then

$$k \circ h = i_X, \qquad h \circ k = i_Y.$$

Applying the theorem,

$$k_* \circ h_* = (k \circ h)_* = (i_X)_* = i_*,$$

and

$$h_* \circ k_* = (h \circ k)_* = (i_Y)_* = j_*$$

where i and j are the identity maps on (X, x_0) and (Y, y_0), respectively.
Thus k_* is the inverse of h_*, so h_* is an isomorphism. □

1.2.3 Contractible Spaces and Retractions

Definition 1.2.6. A space X is called *contractible* if the identity map $id_X : X \longrightarrow X$ is nullhomotopic.

Example 1.2.7. I, $\mathbb{R}$, the Euclidean space $\mathbb{R}^n$, the n-dimensional disc D^n, and the one-point space $\{x\}$ are all contractible. S^1 and S^2 are not contractible.

Proposition 1.2.7. *If X is contractible, then $\pi_1(X, x_0)$ is trivial, for any base point $x_0 \in X$.*

Proof. Since X is contractible, the identity map id_X is homotopic to a constant map $e_{x_0} : X \to X$.
Let f be a loop in X based at x_0. Then

$$f = id_X \circ f \simeq c_{x_0} \circ f = e_{x_0},$$

so $[f] = [e_{x_0}]$ in $\pi_1(X, x_0)$. Hence every element of $\pi_1(X, x_0)$ is trivial. □

Theorem 1.2.8. *Let X be a topological space. The following are equivalent:*

1. *Every continuous map $S^1 \to X$ is homotopic to the constant map.*
2. *Every continuous map $S^1 \to X$ extends to a continuous map $B^2 \to X$, where B^2 is the 2-disk with boundary S^1.*
3. *$\pi_1(X, x_0)$ is trivial, for all $x_0 \in X$.*

Proof. We shall prove this theorem in a later section. □

Definition 1.2.8. A subset $A \subset X$ is called a *retract* of X if there is a map

$$r : X \to A; \text{ called the retraction of X onto A,}$$

such that

$$r|_A = \mathrm{id}_A,$$

that is, if $i : A \hookrightarrow X$ is the inclusion map, then

$$r \circ i = \mathrm{id}_A .$$

A subset $A \subset X$ is called a *deformation retract* of X if, in addition,

$$i \circ r \simeq \mathrm{id}_X .$$

Remark 1.2.9. If A is a deformation retract of X, then A is homotopy equivalent to X.

Lemma 1.2.9. *The n-sphere S^n is a deformation retract of* $\mathbb{R}^{n+1} \setminus \{0\}$.

Proof. Let

$$r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$$

be defined by

$$r(x) = \frac{x}{\|x\|}.$$

Then $r|_{S^n} = \mathrm{id}_{S^n}$, so r is a retraction.
Also, if $i : S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is the inclusion map, define

$$H : (\mathbb{R}^{n+1} \setminus \{0\}) \times I \to \mathbb{R}^{n+1} \setminus \{0\}$$

by

$$H(x, t) = \frac{x}{(1-t) + t\|x\|}.$$

Then H is continuous, and

$$H(x, 0) = x = \mathrm{id}_X(x), \qquad H(x, 1) = \frac{x}{\|x\|} = (i \circ r)(x).$$

Hence $i \circ r \simeq \mathrm{id}_X$, and so S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$. □

1.3 The Fundamental Group of the Circle

In the previous sections, we introduced the fundamental group and established its basic algebraic properties. We now turn to the computation of the fundamental group of the circle, one of the central examples in algebraic topology. The main tool for this computation is the theory of *covering spaces.*
In this section, we introduce only the concepts and results on covering spaces that are required for the computation of $\pi_1(S^1)$. A more systematic treatment of covering spaces, together with their deeper properties and applications, will follow later in these notes.

1.3.1 Covering Spaces

Definition 1.3.1. Let
$$p : E \longrightarrow B$$
be a continuous surjective map.
An open subset $U \subseteq B$ is said to be *evenly covered* by p if
$$p^{-1}(U)$$
can be written as the union of pairwise disjoint open subsets
$$\{V_\alpha\}_{\alpha\in J}$$
of E such that, for each α,
$$p|_{V_\alpha} : V_\alpha \longrightarrow U$$
is a homeomorphism.
The collection $\{V_\alpha\}$ is called a *partition of* $p^{-1}(U)$ *into slices*, and each V_α is called a *slice* over U.

Geometrically, if U is evenly covered by p, then $p^{-1}(U)$ consists of several disjoint copies of U, each mapped homeomorphically onto U by p. One may think of these copies as a stack of identical "sheets" lying above U.

Definition 1.3.2. A continuous surjective map
$$p : E \longrightarrow B$$
is called a *covering map* if every point $b \in B$ has an open neighbourhood U that is evenly covered by p. In this case, E is called a *covering space* of B.

Definition 1.3.3. Let
$$f : X \to Y$$
be a continuous map.
We say that f is a *local homeomorphism* if for every point $x \in X$, there exists an open neighbourhood U of x such that $f(U)$ is open in Y and the restriction
$$f|_U : U \to f(U)$$
is a homeomorphism.

Remark 1.3.4. Every covering map is a local homeomorphism. Indeed, if $x \in E$ and U is an evenly covered neighbourhood of $p(x)$, then the slice containing x is mapped homeomorphically onto U. The converse is not true in general.

Example 1.3.5. Let X be any topological space. The identity map

$$\mathrm{id}_X : X \longrightarrow X$$

is a covering map.
More generally, let

$$E = X \times \{1, \dots, n\},$$

and define

$$p : E \longrightarrow X, \qquad p(x,i) = x.$$

Then p is a covering map. Here every point of X has exactly n distinct preimages.

Theorem 1.3.1. *The map*

$$p : \mathbb{R} \longrightarrow S^1, \qquad p(x) = (\cos 2\pi x, \sin 2\pi x),$$

is a covering map.

Proof. The map p wraps the real line around the circle, sending each interval

$$[n, n+1]$$

onto S^1.
To verify the covering property, consider the open subset

$$U = \{(x,y) \in S^1 : x > 0\}.$$

Then

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4} \right).$$

For each

$$V_n = \left(n - \frac{1}{4}, n + \frac{1}{4} \right),$$

the restriction

$$p|_{V_n} : V_n \longrightarrow U$$

is a homeomorphism. Since the sets V_n are pairwise disjoint and their union is $p^{-1}(U)$, the set U is evenly covered.
Similarly, the open subsets obtained by intersecting S^1 with the upper half-plane, the lower half-plane, and the left half-plane are evenly covered. These open sets cover S^1.
Hence every point of S^1 has an evenly covered neighbourhood, so

$$p : \mathbb{R} \longrightarrow S^1$$

is a covering map. □

1.3.2 The Fundamental Group of the Circle

Definition 1.3.6. Let

$$p : E \to B$$

be a map. If f is a continuous mapping of a space X into B, a *lifting* of f is a map

$$\widetilde{f} : X \to E$$

such that

$$p \circ \widetilde{f} = f.$$

Lemma 1.3.2 (Path Lifting)**.** *Let*

$$p : E \to B$$

be a covering map, let $p(e_0) = b_0$*, and let*

$$f : I \to B$$

be any path beginning at b_0*. Then* f *has a unique lifting*

$$\widetilde{f} : I \to E$$

beginning at e_0*.*

Proof. Cover B by open sets that are evenly covered by p. Subdivide I into finitely many subintervals so that the image of each subinterval lies in one of these open sets. Define the lifting step by step on each subinterval, using the fact that on every slice of an evenly covered set, p is a homeomorphism.
Existence follows from this construction. Uniqueness is proved inductively on the subintervals, since on each slice there is only one possible lift once the initial point is fixed. □

Lemma 1.3.3 (Homotopy Lifting)**.** *Let*

$$p : E \to B$$

be a covering map, let $p(e_0) = b_0$*, and let*

$$F : I \times I \to B$$

be continuous with

$$F(0, 0) = b_0.$$

Then there is a unique lifting

$$\widetilde{F} : I \times I \to E$$

such that

$$\widetilde{F}(0, 0) = e_0.$$

If F *is a path homotopy, then* $\widetilde{F}$ *is also a path homotopy.*

Proof. The lifting is constructed rectangle by rectangle. First one lifts the edges $I \times \{0\}$ and $\{0\} \times I$, and then one extends across successive small rectangles whose images lie in evenly covered open sets.
At each stage, the lift is uniquely determined by the values on the boundary already constructed. The pasting lemma gives continuity of the resulting map.
If F is a path homotopy, then the lifted map sends the two vertical edges of $I \times I$ into the single point set $p^{-1}(b_0)$, which is discrete as a subspace of E. Since $I \times I$ is connected, each of these edges must map to a single point. Hence the lifting is again a path homotopy. □

Theorem 1.3.4. *Let*

$$p : E \to B$$

be a covering map, let $p(e_0) = b_0$*, and let* f *and* g *be paths in* B *from* b_0 *to* b_1*. Let* $\widetilde{f}$ *and* $\widetilde{g}$ *be their respective liftings to paths in* E *beginning at* e_0*. If* f *and* g *are path homotopic, then* $\widetilde{f}$ *and* $\widetilde{g}$ *end at the same point of* E *and are path homotopic.*

Proof. Let $F : I \times I \to B$ be a path homotopy between f and g. By the preceding lemma, F has a lifting

$$\widetilde{F} : I \times I \to E$$

with $\widetilde{F}(0,0) = e_0$.
Since F is a path homotopy, the lifted map $\widetilde{F}$ is also a path homotopy. The restrictions of $\widetilde{F}$ to the bottom and top edges of $I \times I$ are liftings of f and g beginning at e_0. By uniqueness of path lifting, these must be $\widetilde{f}$ and $\widetilde{g}$. Therefore the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree, and the two paths are path homotopic. □

Definition 1.3.7. Let

$$p : E \to B$$

be a covering map and let $b_0 \in B$. Choose $e_0 \in E$ such that $p(e_0) = b_0$.
Given an element $[f] \in \pi_1(B, b_0)$, let $\widetilde{f}$ denote the lifting of f to a path in E beginning at e_0. Define

$$\phi([f]) = \widetilde{f}(1).$$

Then ϕ is a well-defined map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

We call ϕ the *lifting correspondence* derived from the covering map p.

Theorem 1.3.5. *Let*

$$p : E \to B$$

be a covering map, let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof. If E is path connected, then given $e_1 \in p^{-1}(b_0)$, there is a path $\widetilde{f}$ in E from e_0 to e_1. Then $f = p \circ \widetilde{f}$ is a loop in B at b_0, and by definition $\phi([f]) = e_1$.
Now assume that E is simply connected. Let $[f]$ and $[g]$ be elements of $\pi_1(B, b_0)$ such that

$$\phi([f]) = \phi([g]).$$

Let $\widetilde{f}$ and $\widetilde{g}$ be the liftings of f and g respectively, beginning at e_0. Since E is simply connected, $\widetilde{f}$ and $\widetilde{g}$ are path homotopic. Hence $p \circ \widetilde{F}$ is a path homotopy in B between f and g. Thus $[f] = [g]$.
Therefore ϕ is bijective. □

Theorem 1.3.6. *The fundamental group of S^1 is isomorphic to the additive group of integers:*

$$\pi_1(S^1, s_0) \cong \mathbb{Z}.$$

Proof. Let

$$p : \mathbb{R} \to S^1, \qquad p(x) = (\cos 2\pi x, \sin 2\pi x)$$

be the covering map of Theorem 53.1, and let $e_0 = 0$, $b_0 = p(e_0)$.
Since $\mathbb{R}$ is simply connected, the lifting correspondence

$$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is bijective.

It remains to show that ϕ is a homomorphism. Given $[f]$ and $[g]$ in $\pi_1(S^1, b_0)$, let $\widetilde{f}$ and $\widetilde{g}$ be their liftings beginning at 0, and let

$$\phi([f]) = n, \qquad \phi([g]) = m.$$

Define a path $\widetilde{g}'$ in $\mathbb{R}$ by

$$\widetilde{g}'(s) = n + \widetilde{g}(s).$$

Then $\widetilde{g}'$ is a lifting of g beginning at n. The concatenation

$$\widetilde{f} * \widetilde{g}'$$

is a lifting of $f * g$ beginning at 0, and its endpoint is $n + m$. Hence

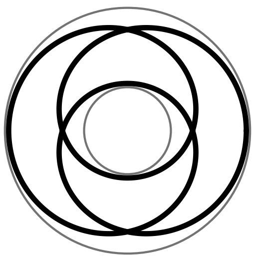
$$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g]).$$

Thus ϕ is an isomorphism, and the theorem follows. $\square$

1.4 Applications of the Fundamental Group

We now apply the theory developed in the previous sections to obtain several classical results concerning retractions and nullhomotopic maps but first we have the following example as an application of Proposition 1.2.4;

Example 1.4.1. We have an isomorphism $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. Under this isomorphism a pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to a loop that winds p times around one S^1 factor of the torus and q times around the other S^1 factor, for example the loop $\omega_{pq}(s) = (\omega_p(s), \omega_q(s))$, where $\omega_n = (cos2\pi ns, sin2\pi ns)$. Interestingly, this loop can be knotted, as seen in figure below for the case $p = 3$, $q = 2$. The knots that arise in this fashion are called torus knots.



More generally, the n-dimensional torus, which is the product of n circles, has fundamental group isomorphic to the product of n copies of $\mathbb{Z}$. This follows by induction on n.

1.4.1 Brouser fixed-point and Borsuk-Ulam Theorems

First we recall the definition of a retract. If $A \subset X$, a *retraction* of X onto is a continuous map $r : X \longrightarrow A$ such that $r|_A$ is the identity of map A. The map r is called a *retract* of X onto A.

Lemma 1.4.1. *If A is a retract of X, then the homomorphism of fundamental groups induced by the inclusion*

$$j : A \hookrightarrow X$$

is injective.

Proof. Let
$$r : X \to A$$
be a retraction. Then
$$r \circ j = \mathrm{id}_A .$$
Applying the induced homomorphism gives
$$r_* \circ j_* = (r \circ j)_* = \mathrm{id}_{\pi_1(A)} .$$
Hence j_* must be injective. □

Theorem 1.4.2 (No-Retraction Theorem)**.** *There is no retraction of B^2 onto S^1.*

Proof. Suppose, to the contrary, that
$$r : B^2 \to S^1$$
is a retraction. By the preceding lemma, the inclusion
$$j : S^1 \hookrightarrow B^2$$
induces an injective homomorphism
$$j_* : \pi_1(S^1) \to \pi_1(B^2).$$
However,
$$\pi_1(S^1) \cong \mathbb{Z}, \qquad \pi_1(B^2) \text{ is trivial,}$$
which is impossible. Therefore no such retraction exists. □

Now, we prove theorem 1.2.8. First, we state it as a lemma with slight change of language.

Lemma 1.4.3. *Let $h : S^1 \longrightarrow X$ be a continuous map. Then the following are equivalent.*

1. *h is nullhomotopic.*
2. *h extends to a continuous map $k : B^2 \longrightarrow X$.*
3. *The induced homomorphism*
$$h_* : \pi_1(S^1, b_0) \longrightarrow \pi_1(X, h(b_0))$$
is trivial.

Proof. **(1)⇒(2).** Suppose $H : S^1 \times I \to X$ is a homotopy between h and a constant map. Let $\pi : S^1 \times I \to B^2$ be given as
$$\pi(x, t) = (1 - t)x.$$
The map π is continuous, closed, and surjective, so it is a quotient map that collapses $S^1 \times \{1\}$ to 0 and is otherwise injective. Since H is constant on $S^1 \times \{1\}$, it induces, via the qoutient map π a continuous map
$$k : B^2 \to X$$
satisfying
$$h = k|_{S^1}.$$

$$\begin{tikzcd} S^1 \times I \arrow[r, "H"] \arrow[dr, "\pi"'] & X \\ & B^2 \arrow[u, dashed, "k"'] \end{tikzcd}$$

(2)⇒(3). Let

$$j : S^1 \hookrightarrow B^2$$

be the inclusion. Then

$$h = k \circ j,$$

we have

$$h_* = k_* \circ j_*.$$

Because $\pi_1(B^2) = 0$, the homomorphism j_* is trivial. Hence h_* is trivial.

(3)⇒(1). Let $p : \mathbb{R} \to S^1$ be the standard covering map, and let $p_0 : \mathbb{R} \to S^1$ be its restriction to unit interval. Then $[p_0]$ generates $\pi_1(S^1, b_0)$ because p_0 is a loop in S^1 whose lift to $\mathbb{R}$ begins at 0 and ends at 1. Let $x_0 = h(b_0)$. Since h_* is trivial,

$$[f] = [h \circ p_0]$$

is the identity element of

$$\pi_1(X, x_0).$$

Thus $h \circ p_0$ is nullhomotopic.
Using the quotient map

$$p_0 \times \mathrm{id} : I \times I \to S^1 \times I,$$

this homotopy induces a homotopy between h and a constant map: For if F is a homotopy in X between f and the constant path at x_0 then F maps $0 \times I, 1 \times I,$ and $I \times 1$ to x_0, whereas $p_0 \times id$ maps $0 \times t, 1 \times t$ to $b_0 \times t$ for each t, and is otherwise injective. Hence h is nullhomotopic.

$$\begin{tikzcd} I \times I \arrow[r, "F"] \arrow[dr, "p_0 \times id"'] & X \\ & B^2 \arrow[u, dashed, "H"'] \end{tikzcd}$$

□

Corollary 1.4.4. *The inclusion map*

$$j : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$$

is not nullhomotopic. The identity map

$$i : S^1 \to S^1$$

is not nullhomotopic.

Proof. There is a retraction

$$r : \mathbb{R}^2 \setminus \{0\} \to S^1 \qquad s.t \qquad r(x) = \frac{x}{\|x\|}.$$

Hence the induced homomorphism

$$j_* : \pi_1(S^1, b_0) \to \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$$

is injective, and therefore nontrivial. By the preceding lemma, the inclusion j cannot be nullhomotopic.
Similarly, i_* is the identity homomorphism, so the identity map on S^1 is not nullhomotopic. □

Definition 1.4.2 (Vector Field)**.** A *vector field* on B^2 is an ordered pair $(x, v(x))$, where $x \in B^2$ and

$$v : B^2 \longrightarrow \mathbb{R}^2$$

is a continuous map.
In calculus, one often uses the notation

$$v(x) = v_1(x)\mathbf{i} + v_2(x)\mathbf{j}$$

for the function v, where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit basis vectors in $\mathbb{R}^2$. We shall, however, stick with simple functional notation.
A vector field is said to be *nonvanishing* if

$$v(x) \neq 0$$

for every $x \in B^2$; equivalently, v maps B^2 into $\mathbb{R}^2 - \{0\}$.

Theorem 1.4.5. *Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.*

Proof. Suppose first that $v(x)$ does not point directly inward at any point x of S^1 and derive a contradiction.
Consider the map

$$w : B^2 \longrightarrow \mathbb{R}^2 - \{0\},$$

and let w be its restriction to S^1. Since the map v extends to a map of B^2 into $\mathbb{R}^2 - \{0\}$, it is nullhomotopic.
On the other hand, w is homotopic to the inclusion map

$$j : S^1 \longrightarrow \mathbb{R}^2 - \{0\}.$$

We defines a homotopy between them by the equation

$$F(x, t) = tx + (1 - t)w(x),$$

for $x \in S^1$.
We must show that $F(x, t) \neq 0$. Clearly, $F(x, t) \neq 0$ for $t = 0$ and $t = 1$. If $F(x, t) = 0$ for some t with $0 < t < 1$, then $tx + (1 - t)w(x) = 0$, so that $w(x)$ equals a negative scalar multiple of x. But this means that $w(x)$ points directly inward at x. Hence F maps $S^1 \times I$ into $\mathbb{R}^2 - \{0\}$, as desired.
It follows that j is nullhomotopic, contradicting the preceding corollary.
To show that v points directly outward at some point of S^1, we apply the result just proved to the vector field $(x, -v(x))$. □

Theorem 1.4.6 (Brouwer fixed-point theorem for the disc)**.** *If*

$$f : B^2 \longrightarrow B^2$$

is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.

Proof. We proceed by contradiction.
Suppose that $f(x) \neq x$ for every $x \in B^2$. Then defining

$$v(x) = f(x) - x$$

gives a nonvanishing vector field $(x, v(x))$ on B^2.
But the vector field v cannot point directly outward at any point x of S^1, for that would mean

$$f(x) - x = ax$$

for some positive real number a, so that

$$f(x) = (1 + a)x$$

would lie outside the unit ball B^2. We thus arrive at a contradiction. □

One might well wonder why fixed-point theorems are of interest in mathematics. It turns out that many problems, such as problems concerning existence of solutions for systems of equations, for instance, can be formulated as fixed-point problems. Here is one example, a classical theorem of Frobenius. We assume some knowledge of linear algebra at this point.

Corollary 1.4.7. *Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue (characteristic value).*

Proof. Let

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

be the linear transformation whose matrix (relative to the standard basis for $\mathbb{R}^3$) is A.
Let B be the intersection of the 2-sphere S^2 with the first octant

$$\{(x_1, x_2, x_3) \mid x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$$

of $\mathbb{R}^3$. It is easy to show that B is homeomorphic to the ball B^2, so that the fixed-point theorem holds for continuous maps of B into itself.
Now if $x = (x_1, x_2, x_3)$ is in B, then all the components of x are nonnegative and at least one is positive.
Because all entries of A are positive, the vector $T(x)$ is a vector all of whose components are positive. As a result, the map

$$x \longmapsto \frac{T(x)}{\|T(x)\|}$$

is a continuous map of B to itself, which therefore has a fixed point x_0. Then

$$T(x_0) = \|T(x_0)\| x_0,$$

so that T (and therefore the matrix A) has the positive real eigenvalue $\|T(x_0)\|$. □

Definition 1.4.3. If x is a point of S^m, then its *antipode* is the point $-x$. A map $h : S^n \longrightarrow S^m$ is said to be *antipode-preserving* if $h(-x) = -h(x)$ for all $x \in S^n$

Theorem 1.4.8 (Borsuk-Ulam theorem for S^2)**.** *Given a continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.*

Proof. Suppose, to the contrary, that

$$f(x) \neq f(-x)$$

for every $x \in S^2$. Define $g : S^2 \longrightarrow S^1$ as

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Since

$$g(-x) = \frac{f(-x) - f(x)}{\|f(-x) - f(x)\|} = -g(x),$$

the map g is antipode-preserving.
Let $\eta : I \longrightarrow S^2$ be the loop circling the equator of $S^2 \subset \mathbb{R}^3$ given by

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0),$$

and define $h = g \circ \eta : I \longrightarrow S^1$. Since

$$\eta\left(s + \frac{1}{2}\right) = -\eta(s),$$

we have

$$h\left(s + \frac{1}{2}\right) = g\left(\eta\left(s + \frac{1}{2}\right)\right) = g(-\eta(s)) = -g(\eta(s)) = -h(s)$$

for every $s \in \left[0, \frac{1}{2}\right]$.
By the path lifting property for the covering map

$$p : \mathbb{R} \longrightarrow S^1; \qquad p(t) = e^{2\pi i t},$$

the loop h admits a unique lift

$$\widetilde{h} : I \longrightarrow \mathbb{R}.$$

Since $h(s + \frac{1}{2}) = -h(s)$, there exists an odd integer q such that

$$\widetilde{h}\left(s + \frac{1}{2}\right) = \widetilde{h}(s) + \frac{q}{2}.$$

Indeed, q is independent of s, since

$$q = 2\left(\widetilde{h}\left(s + \frac{1}{2}\right) - \widetilde{h}(s)\right)$$

depends continuously on s, while taking values in the integers. Hence q is constant.
Taking $s = \frac{1}{2}$, we obtain

$$\widetilde{h}(1) = \widetilde{h}\left(\frac{1}{2}\right) + \frac{q}{2} = \widetilde{h}(0) + q.$$

Therefore the loop h represents q times a generator of $\pi_1(S^1)$. Since q is odd, $q \neq 0$, so h is not nullhomotopic.
On the other hand,

$$h = g \circ \eta,$$

and the loop η is nullhomotopic in S^2. Hence h is also nullhomotopic, a contradiction.
Therefore our assumption was false, and there exists $x \in S^2$ such that

$$f(x) = f(-x).$$

$\square$

Corollary 1.4.9. *Whenever S^2 is expressed as the union of three closed sets A_1 , A_2 , and A_3, then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$.*

Proof. Let $d_i : S^2 \longrightarrow \mathbb{R}$ measure distance to A_i, that is, $d_i(x) = inf_{y \in A_i}|x - y|$. This is a continuous function, so we may apply the Borsuk-Ulam theorem to the map $S^2 \longrightarrow R^2$, $x \mapsto (d_1(x), d_2(x))$, obtaining a pair of antipodal points x and $-x$ with $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$. If either of these two distances is zero, then x and $-x$ both lie in the same set A_1 or A_2 since these are closed sets. On the other hand, if the distances from x and $-x$ to A_1 and A_2 are both strictly positive, then x and x lie in neither A_1 nor A_2 so they must lie in A_3. □

1.4.2 The Fundamental Theorem of Algebra

Another applications of the fundamental group is its use in the proof of the Fundamental Theorem of Algebra. The argument below shows how topological ideas can be used to prove that every nonconstant polynomial with complex coefficients has a root.

Lemma 1.4.10. *Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number. Then the induced homomorphism f_* of fundamental groups is injective.*

Proof. Let $p_0 : I \to S^1$ be the standard loop in S^1,

$$p_0(s) = e^{2\pi is} = (\cos 2\pi s, \sin 2\pi s).$$

Its image under f_* is the loop

$$f(p_0(s)) = (e^{2\pi is})^n = (\cos 2\pi ns, \sin 2\pi ns).$$

This loop lifts to the path $s \mapsto ns$ in the covering space $\mathbb{R}$. Therefore, the loop $f \circ p_0$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1, b_0)$ with the integers, whereas p_0 corresponds to the number 1. Thus f_* is "multiplication by n" in the fundamental group of S^1, so that in particular, f_* is injective. □

Lemma 1.4.11. *Let $g : S^1 \to \mathbb{R}^2 - 0$ be the map $g(z) = z^n$, then g is not nullhomotopic.*

Proof. The map g equals the map f of Lemma 1.4.10 followed by the inclusion map $j : S^1 \to \mathbb{R}^2 - 0$. That is, $g = j \circ f$. Now f_* is injective, and j_* is injective because S^1 is a retract of $\mathbb{R}^2 - 0$(see lemma 1.4.1). Therefore, $g_* = j_* \circ f_*$ is injective. Thus g cannot be nullhomotopic. □

Theorem 1.4.12 (The fundamental theorem of algebra)**.** *A polynomial equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

Proof. **Step 1.** First we prove a special case of the theorem. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we assume that

$$|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$$

and show that the equation has a root lying in the unit ball B^2.
Assume it has no such root. Then we can define a map $k : B^2 \to \mathbb{R}^2 - 0$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let h be the restriction of k to S^1. Because h extends to a map of the unit ball into $\mathbb{R}^2 - 0$, the map h is nullhomotopic(lemma 1.4.3).
On the other hand, we shall define a homotopy F between h and the map g of lemma 1.4.11 as; $F : S^1 \times I \to \mathbb{R}^2 - 0$ is defined by the equation

$$F(z,t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0).$$

$F(z,t)$ never equals 0 because

$$|F(z,t)| \geq |z^n| - t|a_{n-1}z^{n-1} + \cdots + a_0|$$

$$\geq 1 - t\left(|a_{n-1}z^{n-1}| + \cdots + |a_0|\right)$$

$$= 1 - t\left(|a_{n-1}| + \cdots + |a_0|\right) > 0.$$

Since g is not nullhomotopic, we arrive at a contradiction.
Step 2. Now we prove the general case. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,$$

let us choose a real number $c > 0$ and substitute $x = cy$. We obtain the equation

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

or

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0.$$

If this equation has the root $y = y_0$, then the original equation has the root $x_0 = cy_0$. So we need merely choose c large enough that

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

to reduce the theorem to the special case considered in Step 1, which completes the proof. □

1.4.3 Homotopy Type

Now, we will discuss another way of obtaining information about fundamental group a space X, involving the notion of homotopy type.

Lemma 1.4.13. *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and if image of the base point x_0 of X remains fixed at y_0 during the homotop, the the homomorphisms h_* and k_* of fundamental groups of X and Y are equal.*

Proof. By assumption, there is a homotopy $H : X \times I \to Y$ between h and k such that $H(x_0, t) = y_0$ for all t.
Let f be a loop in X based at x_0, then $h \circ f$ and $k \circ f$ are loops in Y based at y_0.
We define a map $F : I \times I \to Y$ as

$$F(s,t) = H(f(s), t).$$

Then, clearly F is continuous and

$$F(s,0) = H(f(s), 0) = h(f(s)) = (h \circ f)(s);$$

$$F(s,1) = H(f(s),1) = k(f(s)) = (k \circ f)(s).$$

Hence, F is a homotopy between $h \circ f$ and $k \circ f$. Therefore, h_* and k_* are equal as

$$h_*([f]) = [h \circ f] = [k \circ f] = k_*([f]).$$

□

We now use this lemma to prove the following result:

Theorem 1.4.14. *The inclusion map $j : S^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ induces an isomorphism of fundamental groups.*

Proof. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$, and $b_0 = (1, 0, \ldots, 0)$, and define $r : X \longrightarrow S^n$ as,

$$r(x) = \frac{x}{\|x\|}.$$

Then $r \circ j = \mathrm{id}_{S^n}$, so $r_* \circ j_* = \mathrm{id}_{\pi_1(S^n, b_0)}$.
Now consider the composite $j \circ r : X \longrightarrow X$.
Although this is not the identity map of X, it is homotopic to it. Indeed, define

$$H : X \times I \longrightarrow X$$

by the straight-line homotopy

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}.$$

We can write

$$H(x,t) = \left((1-t) + \frac{t}{\|x\|}\right) x.$$

The coefficient

$$(1-t) + \frac{t}{\|x\|}$$

is positive, so $H(x,t) \neq 0$. Hence H is a homotopy between id_X and $j \circ r$. Moreover, $H(b_0, t) = b_0$ for every $t \in I$, since $\|b_0\| = 1$. Thus the homotopy preserves the base point.
It follows from previous lemma that

$$(j \circ r)_* = j_* \circ r_* = \mathrm{id}_{\pi_1(X, b_0)}.$$

Therefore, $r_* \circ j_* = \mathrm{id}_{\pi_1(S^n, b_0)}$ and $j_* \circ r_* = \mathrm{id}_{\pi_1(X, b_0)}$. Hence j_* and r_* are inverse isomorphisms, and the result follows. □

The preceding result suggests a more general situation in which the same procedure applies. We have already encountered this concept earlier, but we recall its definition here in the present context.

Definition 1.4.4. Let A be a subspace of X. We say that A is a *deformation retract* of X of the identity map of X is homotopic to a map that carries all of X into A, such that every point of A remains fixed during the homotopy. That is, there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a \ \forall \ a \in A$.

The homotopy H is called *deformation retraction* of X onto A. The map $r : X \to A$ is defined as $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between id_X and the map $j \circ r$, where $j : A \to X$ is the inclusion map.
As a consequence, we have the following generalisation of Theorem 1.4.14:

Theorem 1.4.15. *Let A be a deformation retract of X, and let $x_0 \in A$. Then the inclusion map $j : (A, x_0) \longrightarrow (X, x_0)$ induces an isomorphism of fundamental groups.*

Proof. The proof follows as a generalisation of proof of Theorem 1.4.14. We have, A is deformation retract of X, then there exists $r : X \to A$ that collapses X into A, such that r is homotopic to id_X.
Clearly, $r \circ j = id_A$ and thus $r_* \circ j_* = id_{\pi_1(A,x_0)}$.
Now, consider the composite $j \circ r : X \to X$. Then, it is homotopic to id_X via the straight line homotopy $H : X \times I \to X$ such that $H(x,t) = (1-t)x + tr(x)$. Also, $H(x_0,t) = (1-t)x_0 + tr(x_0) = (1-t)x_0 + tx_0 = x_0$, as $r : X \to A$ is a retraction of X onto A. Thus, by lemma 1.4.13, $(j \circ r)_* = j_* \circ r_* = id_{\pi_1(X,x_0)}$, which completes the proof. □