# Review of Research Article "Ranks and Presentaions for Order-Preserving Transformations with one Fixed Point" by Joerg Koppitz and Somuek Worawiset.

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**INTRODUCTION**

The Paper "Ranks and Presentations for Order-Preserving Transformations with one fixed point" investigates the Semigroup $O_{n,p}$, which consists of all order preserving transformations on n-element set $X_n$ with exactly one fixed point $p \in X_n$.
In this paper, the Author addresses two key aspects: determining the rank of the semigroup $O_{n,p}$ for p>1 and providing a semigroup presentation for $O_{n,1}$, of all order-preserving tansformations with $x\alpha = x$ iff $x = 1$ ; $\forall\ \alpha \in O_{n,1}$ and $x \in X_n$.
By examining these elements, the paper enhances our understanding of the structure and properties of these semigroups within the broader context of the Transformation Semigroup $T_n$.

**NOTATIONS**

- We shall use the symbol $O_n$ to denote the semigroup of all Order-Preserving Transformations on $X_n$.
- $Fix(\alpha) = \{x \in X_n : x\alpha = x\}$.
- $O_{n,A} = \{\alpha \in O_n : Fix(\alpha) = A\}$.
- $G_{n,p} = \{\alpha \in O_{n,p} : |x\alpha - x| = 1\, for\, atleast\, one\, x \in X_n \backslash \{p-1, p-2\}\}$.
- If Y is any set then we denote by $Y^+$ the free semigroup on Y.
- If $R \subseteq Y^+ \times Y^+$, then we denote by $R^\#$ the congruence on $Y^+$ generated by R.
- We denote by $A_n$ the set of all mappings $g \in \{0, 1, 2, \ldots, n-1\}^{X_n}$ (i.e, $g : X_n \longrightarrow \{0, 1, 2, \ldots, n-1\}$) with 1g=0, xg$\geq$1 for x $\in \{2, 3, \ldots, n\}$, where lg = 1 for some l $\in \{3, 4, \ldots, n\}$, and either (x+1)g $\leq$ xg or (x+1)g = xg+1 $\forall$ x $\in \{1, 2, \ldots, n-1\}$
- For $g, h \in A_n, let\, g \oplus h : X_n \longrightarrow \{0, 1, \ldots, n-1\}\ with\ 1(g \oplus h) = 0,\ 2(g \oplus h) = 1\, and\, x(g \oplus h) = (x - xg)h + xg - 1\, for\, x \in \{3, 4, \ldots, n\}$.

- For each g $\in A_n$, let $\alpha_g \in T_n$ with

$$x\alpha_g = x - xg \;\forall\; x \in X_n.$$

**KEY CONCEPTS**

- **Order-preserving Transformation:** A transformation $\alpha$ is order preserving if $x < y$ implies $x\alpha \leq y\alpha$.

- **Fixed Point:** If $\alpha$ is a transformation on $X_n$ and $x \in X_n$, then x is said to be a fixed point of $\alpha \iff x\alpha = x$.

- **Rank of Semigroup:** The rank of a semigroup S is the cardinality of the smallest generating set of S. i.e, $rank(S) = min\{|A| :< A >= S\}$

- $O_{n,p}$ **is a nilpotent semigroup.**

- **Semigroup Presentations:** We say that a Semigroup S has a Semigroup Presentation $< Y|R >$ is to say that $S \cong \frac{Y^+}{R^\#}$.
  Equivalently, we say that there is a semigroup epimorphism $\phi$:$Y^+ \longrightarrow S$ such that $Ker(\phi) = \{(x, y) \in Y^+ \times Y^+ : x\phi = y\phi\} = R^\#$.
  Elements of Y are called generators and elements of R are called relations.
  In practice (x,y) $\in$ R will be written as $x \approx y$.

- We define $w : X_n \longrightarrow \{0, 1, \ldots, n-1\}$ $as$ $1w = 0\, and\, xw = 1 \,\forall\, x \in \{2, 3, \ldots, n\}$. Clearly $w \in A_n$.

- We define a mapping $f : X_n \cup \{0\} \longrightarrow X_n$ by

$$f(0) = 0 \; and \; f(x) = x \;\forall\; x \in X_n.$$

- For $g \in A_n$, let $\hat{g} \in \{0, 1, \ldots, n-1\}^{X_n}$ with 1$\hat{g}$ = 0, 2$\hat{g}$ = 1, and x$\hat{g}$ = $x - 1 - f(f(x - 1 - xg) - 1)$ for $x \in \{3, 4, \ldots, n\}$.

- We define an alphabet set $Y_n = \{x_g : g \in A_n\}$, and define two sets of relations on $Y_n^+$. Let

$$R_1 = \{x_g x_w^2 \approx x_{\hat{g}} x_w : g \in A_n\} \text{ and}$$

$$R_2 = \{x_g x_h \approx x_{g \oplus h} x_w : g, h \in A_n\}.$$

- We define an epimorphism $\phi : Y_n \longrightarrow O_{n,1}$ by $x_g \phi = \alpha_g$ for $g \in A_n$.

**MAIN RESULTS**

1. Proposition 1.5 $rank(O_{n,1}) = rank(O_{n,n}) = c_{n-1}c_{n-2}$.

2. Theorem 2.4 $rank(O_{n,p}) = c_{p-1}c_{n-p} - c_{p-2}c_{n-p-1}$.

3. Corollary 3.3 $\{\alpha_g : g \in A_n\}$ is a generating set of $O_{n,1}$.

4. Theorem 3.11 The Semigroup $O_{n,1}$ has presentation $< Y_n | R_1 \cup R_2 >$ via $\phi$.

**SUMMARY**

In this paper, the main focus is to calculate the $rank(O_{n,p})$ and give a semigroup presentation for $O_{n,1}$. The authors begin by establishing that the Semigroup $O_{n,1}$ is the maximal nilpotent subsemigroup of the catalan monoid $C_n$ of all order-decreasing and Order-Preserving transformations on $X_n$. Building on this foundation, the authors refer to the rank of $O_{n,1}$ as determined in [15] and also demonstrates the nilpotent nature of On,p for p≥2. Further, for p>1, using the result that for a finite nilpotent semigroup S, the set $S \setminus S^2$ is the unique minimal generating set [13][Lemma 2.0.2], the authors conclude that the $rank(O_{n,p}) = |O_{n,p} \setminus O_{n,p}^2|$. Additionally,it is shown that $O_{n,p} \setminus O_{n,p}^2 = G_{n,p}$ and this is used to calculate the cardinality, $|O_{n,p}| = |O_{n-2,p-1}|$. With these in hand, Theorem 2.4 is employed to calculate the $rank(O_{n,p})$ for p>1.
Finally Theorem 3.11 is employed to provided a semigroup presentation for $O_{n,1}$, contributing to a formal description for generators and relations that define the Semigroup $O_{n,1}$. And at the end the authors illustrate theorem 3.11 by taking n = 4 and obtain a specific semigroup presentation for $O_{4,1}$, demonstrating the theorem's utility in a concrete example.
Throug these contributions, the paper enhances our understanding of the algebraic properties of order-preserving transformations with one fixed point.

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